

Partial holomorphic connections and extension of foliations

Isaia Nisoli

Abstract This paper stresses the strong link between the existence of partial holomorphic connections on the normal bundle of a foliation seen as a quotient of the ambient tangent bundle and the extendability of a foliation to an infinitesimal neighborhood of a submanifold. We find the obstructions to extendability, and thanks to the theory developed we obtain some new Khanedani–Lehmann–Suwa type index theorems.

Contents

0. Introduction	517
1. Foliations of k th infinitesimal neighborhoods	520
2. The Atiyah sheaf for the variation action	526
3. Splittings and foliations of the first infinitesimal neighborhood	533
4. Extension of foliations and embedding in the normal bundle	538
5. Action of subsheaves of \mathcal{F} on $\mathcal{N}_{\mathcal{F},M}$	544
6. Singular holomorphic foliations of the first infinitesimal neighborhood	546
7. Index theorems for foliations and involutive closures	548
8. Computing the residue in the simplest case	550
9. The residue for the simplest transversal case	552
10. A couple of remarks about extendability of foliations	553
Acknowledgments	554
References	554

0. Introduction

Localization of characteristic classes is an important tool in differential geometry, topology, and dynamics in particular for complex dynamical systems (see [CS]). In this context many different indexes have been developed during the years, among them the Baum–Bott and the Camacho–Sad indexes. A global framework for this theory has been provided by Suwa and Lehmann (see [Su]): the fundamental principle is that the existence of a flat partial holomorphic connection (called a holomorphic action in [Su]) implies the vanishing of the Chern classes associated to some vector bundles. Suppose that we are working on a compact manifold M and we have a partial holomorphic connection outside an analytic subset Σ of M . We can localize these Chern classes to Σ and, using Poincaré and

Alexander duality, define the residue of the characteristic class at Σ (we refer to [Su]).

Now, at least two different research directions arise: to adapt such a theory to singular manifolds and submanifolds (see [LS1], [LS2]), or to try to develop new vanishing theorems. This paper falls into the second group. As we said, such vanishing theorems arise when we have the existence of partial holomorphic connections; this is the case when we have a holomorphic foliation which leaves a submanifold S invariant. This gives rise to index theorems for $N_{\mathcal{F}|_S}$, the normal bundle of the foliation seen as a quotient of the tangent bundle of the submanifold (Baum–Bott index), N_S , the normal bundle to the submanifold (Camacho–Sad index), and $N_{\mathcal{F}}|_S$, the normal bundle of the foliation seen as a quotient of the ambient tangent bundle restricted to S (Kahnedani–Lehmann–Suwa or variation index; see [KS], [LS2]). The fundamental reference on all these topics is [Su].

The same techniques allow us to prove other index theorems, even if the holomorphic foliation is transverse to the submanifold, such as the index theorem for the bundle $\text{Hom}(\mathcal{F}, N_S)$, which gives rise to the tangential index (see [Ho], [Bru]).

In the last years, a new theory was developed also for endomorphisms of a complex manifold leaving a submanifold pointwise invariant (see [ABT1]) and the case of foliation transverse to a submanifold in the Camacho–Sad and Baum–Bott case (see [ABT2], [Ca], [CL], [CMS]). The key to the existence of partial holomorphic connections is the vanishing of the Atiyah class, a cohomological obstruction to the splitting of a short exact sequence of sheaves of \mathcal{O}_S -modules (see [Ati]). In the paper [ABT2] the Atiyah sheaf for the normal bundle of a submanifold was described in a more concrete way, giving new insights to the problem. Further developments such as [ABT3] showed the strong connection between the existence of partial holomorphic connections for N_S and the “regularity” of the embedding of a subvariety.

In Section 2 of this paper we find a more concrete realization of the Atiyah sheaf for the normal bundle of a foliation seen as a quotient of the ambient tangent bundle and study some sufficient conditions for the existence of a more general variation action. First of all in Section 1 we define what a foliation of the k th infinitesimal neighborhood of a submanifold is and prove some Frobenius-type theorems for such foliations, which give us the possibility of choosing atlases with some particular structure; in these special atlases, it is clear that the existence of a foliation of the first infinitesimal neighborhood is the key to the existence of partial holomorphic connections on the normal bundle of a foliation seen as a quotient of the ambient tangent bundle. Therefore, to generalize the variation index we have to find foliations of the first infinitesimal neighborhood; with this aim we study the problem of how to “project” a transversal foliation to a tangential one using first-order splitting (see Section 3) and how to extend a foliation of a submanifold S to an infinitesimal neighborhood (see Section 4). Moreover, thanks to the new realization of the Atiyah sheaf, we develop in Section 5 a result about noninvolutive subsheaves of \mathcal{T}_S which extend to the first infinitesimal neighborhood. This gives us information about vanishing of the characteristic classes of

the involutive closure of their restriction to S (the smallest involutive subsheaf of \mathcal{T}_S containing it) and some more results regarding the extension problem. Thanks to the machinery developed we can then prove some new index theorems, generalizing the Khanedani–Lehmann–Suwa action, and compute their indexes in some simple cases.

Notation and conventions. In this paper we are going to use the Einstein summation convention. To ease the understanding of the computations the indexes are going to have a fixed range. In this paper, M is an n -dimensional complex manifold, S is a complex submanifold of codimension m (unless otherwise stated), and \mathcal{F} is a dimension l holomorphic foliation of either M, S or an infinitesimal neighborhood of S , with $l \leq n - m$. Then the indexes have the following range:

- h, k will range in $1, \dots, n$; these are the indexes relative to the coordinate system of M ;
- p, q will range in $m + 1, \dots, n$, in an atlas adapted to S (see Definition 0.1); these are the indexes relative to the coordinates along S ;
- r, s will range in $1, \dots, m$, in an atlas adapted to S ; these are the indexes relative to the coordinates normal to S ;
- i, j will range in $m + 1, \dots, m + l$, in an atlas adapted to \mathcal{F} (see Definition 1.8); these are the indexes relative to the coordinates along \mathcal{F} ;
- u, v will range in $1, \dots, m, m + l + 1, \dots, n$, in an atlas adapted to \mathcal{F} ; these are the the indexes relative to the coordinates normal to \mathcal{F} .

In case we need more indexes of each type, we shall indicate them with a prime ' or put a subscript, for example, r_1 . We shall denote by \mathcal{O}_M the structure sheaf of holomorphic functions on M , by \mathcal{I}_S the ideal sheaf of a submanifold S , and by \mathcal{I}_S^k its k th power as an ideal. If f is an element of \mathcal{O}_M we will denote by $[f]_{k+1}$ its image in $\mathcal{O}_{S(k)} := \mathcal{O}_M / \mathcal{I}_S^{k+1}$. Moreover, we denote by \mathcal{T}_M and \mathcal{T}_S the tangent sheaves to M and S , respectively, where defined. The following are some definitions we will use through the whole paper.

DEFINITION 0.1

Let \mathcal{U} be an atlas for M . We say that \mathcal{U} is *adapted* to S if on each coordinate neighborhood $(U_\alpha, z_\alpha^1, \dots, z_\alpha^n)$ such that $U \cap S$ is not empty, we have that $S \cap U_\alpha = \{z_\alpha^1 = \dots = z_\alpha^m = 0\}$, where m is the codimension of S .

DEFINITION 0.2

Suppose that Z is a complex manifold and \mathcal{O}_Z is the sheaf of holomorphic functions; a *coherent sheaf* is a sheaf \mathcal{S} of \mathcal{O}_Z -modules such that, for every $z \in Z$, there exists a neighborhood U of z and an exact sequence

$$\mathcal{O}_{Z,U}^p \rightarrow \mathcal{O}_{Z,U}^q \rightarrow \mathcal{S}|_U \rightarrow 0$$

for some p - and q -integers.

Equivalently \mathcal{S} is coherent if

(1) it is locally finitely generated; that is, for every point x there exists an open set U and a finite number of sections $s_1, \dots, s_q \in \mathcal{S}(U)$ such that for every y in U the stalk \mathcal{S}_y is generated by $s_1(y), \dots, s_q(y)$;

(2) the sheaf of relations, that is, $\{(f_1, \dots, f_p) \in \mathcal{O}_{Z,U}^p \mid \sum_{i=1}^p f_i s_i = 0\}$, is finitely generated.

For a coherent sheaf \mathcal{S} we define the *singular set* of \mathcal{S} to be

$$\text{Sing}(\mathcal{S}) = \{x \in Z \mid \mathcal{S}_x \text{ is not } \mathcal{O}_Z\text{-free}\}.$$

Since \mathcal{S} is locally free outside $Z \setminus \text{Sing}(\mathcal{S})$, we define the *rank* of \mathcal{S} to be the rank of its restriction to $Z \setminus \text{Sing}(\mathcal{S})$.

1. Foliations of k th infinitesimal neighborhoods

In this section we define and develop a theory for foliations of k th infinitesimal neighborhoods. We use the notion of logarithmic vector fields, introduced in [Sa]. The sheaf of these vector fields is locally free if S is a submanifold.

DEFINITION 1.1

The *k th infinitesimal neighborhood* of a complex submanifold S is the ringed space $(S, \mathcal{O}_{S(k)})$, where by $\mathcal{O}_{S(k)}$ we denote the quotient sheaf $\mathcal{O}_M/\mathcal{I}_S^{k+1}$.

DEFINITION 1.2

A section v of \mathcal{T}_M is called *logarithmic* if $v(\mathcal{I}_S) \subseteq \mathcal{I}_S$. The sheaf $\mathcal{T}_M(\log S) := \{v \in \mathcal{T}_M \mid v(\mathcal{I}_S) \subseteq \mathcal{I}_S\}$ is called the *sheaf of logarithmic sections* and is a subsheaf of \mathcal{T}_M .

The *tangent sheaf of the k th infinitesimal neighborhood*, denoted by $\mathcal{T}_{S(k)}$, is the image of the sheaf homomorphism $\mathcal{T}_M(\log S) \otimes_{\mathcal{O}_M} \mathcal{O}_{S(k)} \rightarrow \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{S(k)}$ and is a sheaf on S . We will say that a section $v \in \mathcal{T}_{S(k)}$ is *tangential to the k th infinitesimal neighborhood*.

Given a subsheaf \mathcal{E} of $\mathcal{T}_{S(k)}$ we define its *restriction to S* , denoted by $\mathcal{E}|_S$, by $\mathcal{E}|_S := \mathcal{E} \otimes \mathcal{O}_S$.

REMARK 1.3

If a point x does not belong to S , the stalk $\mathcal{T}_M(\log S)_x$ coincides with $\mathcal{T}_{M,x}$. Suppose that we have an atlas adapted to S ; if $x \in S$ the stalk $\mathcal{T}_M(\log S)_x$ is generated by

$$z^r \frac{\partial}{\partial z^s}, \frac{\partial}{\partial z^p}.$$

Then a section v of $\mathcal{T}_{S(k)}$ is written locally as

$$v = [a^r]_{k+1} \frac{\partial}{\partial z^r} + [a^p]_{k+1} \frac{\partial}{\partial z^p},$$

where the a^r belong to \mathcal{I}_S .

REMARK 1.4

In the following, given a section v of $\mathcal{T}_{S^{(k)}}$ and an open set U_α of M intersecting S , we denote by \tilde{v}_α a local extension of v to U_α as a section of $\mathcal{T}_M(U_\alpha)$; given an atlas adapted to S it is possible to build such an extension on each coordinate chart. If the open set is clear from the discussion we shall denote the extension simply by \tilde{v} ; please note that such an extension is not only a section of $\mathcal{T}_M(U_\alpha)$ but also a section of $\mathcal{T}_M(\log S)(U_\alpha)$. Taken as an extension \tilde{v} , denoted by $[1]_{k+1}$ the class of 1 in $\mathcal{O}_{S^{(k)}}(U_\alpha)$, we shall denote its restriction to the k th infinitesimal neighborhood by

$$\tilde{v} \otimes [1]_{k+1}.$$

We prove in Lemma 1.5 that this notation is consistent with the fact that the sections of $\mathcal{T}_{S^{(k)}}$ act as derivations of $\mathcal{O}_{S^{(k)}}$. Moreover, given two open sets U_α and U_β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$ and taking two extension \tilde{v}_α and \tilde{v}_β , respectively, it follows from the definition that on $U_\alpha \cap U_\beta$ we have the following equivalence:

$$(1) \quad v = \tilde{v}_\alpha \otimes [1]_{k+1} = \tilde{v}_\beta \otimes [1]_{k+1}.$$

LEMMA 1.5

The sections of $\mathcal{T}_{S^{(k)}}$ act as derivations of $\mathcal{O}_{S^{(k)}}$. Furthermore, given two sections v, w of $\mathcal{T}_{S^{(k)}}$, their bracket, defined on each coordinate patch U_α such that $U_\alpha \cap S \neq \emptyset$ as

$$[v, w] := [\tilde{v}_\alpha, \tilde{w}_\alpha] \otimes [1]_{k+1},$$

where the bracket on the right-hand side is the usual bracket on \mathcal{T}_M , is a well-defined section of $\mathcal{T}_{S^{(k)}}$.

Proof

Let v be a section of $\mathcal{T}_{S^{(k)}}$, and let f be a section of $\mathcal{O}_{S^{(k)}}$. Let U_α and U_β be two coordinate patches of an atlas adapted to S such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$. On U_α we take representatives \tilde{f}_1 and \tilde{f}_2 of f and an extension \tilde{v}_α of v . We define

$$v(f) := \tilde{v}_\alpha(\tilde{f}_1) \otimes [1]_{k+1} = [\tilde{v}_\alpha(\tilde{f}_1)]_{k+1}.$$

Using the fact that \tilde{v}_α is logarithmic it is easily shown that it does not depend on the extension chosen for f . Since by definition the difference of two extensions \tilde{v}_α and \tilde{v}'_α of v is of the form $g_\alpha^h w_{h,\alpha}$ with $g_\alpha^h \in \mathcal{I}_S^{k+1}$ for each $h = 1, \dots, n$, this derivation does not depend on the extension of v chosen. This implies also that if we take extensions \tilde{v}_α and \tilde{v}_β and representatives \tilde{f}_α and \tilde{f}_β for f on U_α and U_β , respectively, we have that on $U_\alpha \cap U_\beta$, the derivation is well defined.

We prove now that the bracket is well defined; if u and v are sections of $\mathcal{T}_{S^{(k)}}$, the bracket is

$$[u, v] = [\tilde{u}, \tilde{v}] \otimes [1]_{k+1}.$$

If \tilde{u}_1, \tilde{u}_2 are two extensions of u and \tilde{v}_1, \tilde{v}_2 are two extension of v , then

$$\begin{aligned} [\tilde{u}_1, \tilde{v}_1] - [\tilde{u}_2, \tilde{v}_2] &= [\tilde{u}_1, \tilde{v}_1] - [\tilde{u}_1, \tilde{v}_2] + [\tilde{u}_1, \tilde{v}_2] - [\tilde{u}_2, \tilde{v}_2] \\ &= [\tilde{u}_1, \tilde{v}_1 - \tilde{v}_2] + [\tilde{u}_1 - \tilde{u}_2, \tilde{v}_2]. \end{aligned}$$

As above, we have

$$\tilde{u}_1 - \tilde{u}_2 = g_\alpha^h w_{h,\alpha}, \quad \tilde{v}_1 - \tilde{v}_2 = t_\alpha^h w_{h,\alpha},$$

with $g_\alpha^h, t_\alpha^h \in \mathcal{I}_S^{k+1}$ for every h . Then

$$\begin{aligned} (2) \quad [\tilde{u}_1, \tilde{v}_1 - \tilde{v}_2] + [\tilde{u}_1 - \tilde{u}_2, \tilde{v}_2] &= [\tilde{u}_1, t_\alpha^h w_{h,\alpha}] + [g_\alpha^h w_{h,\alpha}, \tilde{v}_2] \\ &= \tilde{u}_1(t_\alpha^h)w_{h,\alpha} + t_\alpha^h[\tilde{u}_1, w_{h,\alpha}] \\ &\quad - \tilde{v}_2(g_\alpha^h)w_{h,\alpha} + g_\alpha^h[w_{h,\alpha}, \tilde{v}_2]. \end{aligned}$$

Since both \tilde{v}_1 and \tilde{u}_2 are logarithmic, the restriction to the k th infinitesimal neighborhood of (2) is zero. \square

Therefore, the following definition makes sense.

DEFINITION 1.6

A *regular foliation* of $S(k)$ is a rank l coherent subsheaf \mathcal{F} of $\mathcal{T}_{S(k)}$, such that:

- for every $x \in S$ the stalk $\mathcal{T}_{S(k)}/\mathcal{F}_x$ is $\mathcal{O}_{S(k),x}$ -free;
- for every $x \in S$ we have $[\mathcal{F}_x, \mathcal{F}_x] \subseteq \mathcal{F}_x$ (where the bracket is the one defined in Lemma 1.5);
- the restriction of $\mathcal{F}|_S$ is a rank l foliation of S .

REMARK 1.7

The third condition is a simplifying condition: in the paper [Br] a lot of work is devoted to clarifying and explaining the concept of extension of a foliation, and our definition is a particular case.

The main tool of this section is the holomorphic Frobenius theorem, whose statement can be found, for example, in [Su, pp. 38–42]. Lemma 1.10 is a tool we use in proving the Frobenius theorem for foliations of the k th infinitesimal neighborhood; we give the proof after a definition.

DEFINITION 1.8

Let \mathcal{F} be a rank l regular foliation of S . We say that an atlas $\{(U_\alpha, z_\alpha^1, \dots, z_\alpha^n)\}$ is *adapted to S and \mathcal{F}* if

- $U_\alpha \cap S = \{z_\alpha^1 = \dots = z_\alpha^m = 0\}$,
- $\mathcal{F}|_{U_\alpha \cap S}$ is generated by $\partial/\partial z_\alpha^{m+1}|_S, \dots, \partial/\partial z_\alpha^{m+l}|_S$.

REMARK 1.9

The existence of such an atlas follows from the holomorphic Frobenius theorem cited above.

LEMMA 1.10

Every regular foliation \mathcal{F} of $S(k)$ admits a local frame which can be extended locally by commuting vector fields; that is, for every point $x \in S$ there exists a neighborhood U_x of x in M and commuting sections $\tilde{w}_{m+1}, \dots, \tilde{w}_{m+l}$ of \mathcal{T}_M on U_x such that $w_i := \tilde{w}_i \otimes [1]_{k+1}$ are generators of $\mathcal{F}(U_x \cap S)$.

Proof

Let x be a point of S . We take a coordinate patch (U, ϕ) centered in x , adapted to S and $\mathcal{F}|_S$. Let $\{v_i\}$ be a system of generators of \mathcal{F} in $U \cap S$, and let $\{\tilde{v}_i\}$ be vector fields extending them. Call \mathcal{D} the distribution spanned by the \tilde{v}_i 's. We complete the frame $\{\tilde{v}_i\}$ to a frame $\{\tilde{v}_k\}$ of \mathcal{T}_M , taking as \tilde{v}_t the coordinate fields $\partial/\partial z^t$. Now, we choose holomorphic functions f_i^k such that:

$$\tilde{v}_k = f_k^h \frac{\partial}{\partial z^h}.$$

Please note that the matrix $A := (f_k^h)$ is a matrix of holomorphic functions acting on the right:

$$[\tilde{v}^1, \dots, \tilde{v}^n] = \left| \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right| \cdot A.$$

By hypothesis we know that A is nonsingular in x , so there exists a neighborhood (still denoted by U) of x such that this matrix is invertible with inverse a matrix of holomorphic functions. Let (g_h^k) be its inverse matrix. We define $\tilde{w}_i = g_i^j \tilde{v}_j$, and we denote it by $w_i := \tilde{w}_i \otimes [1]_{k+1}$. Each one of the \tilde{w}_i 's belongs to the module generated by $\tilde{v}_{m+1}, \dots, \tilde{v}_{m+l}$; therefore each w_i belongs to $\mathcal{T}_{S(k)}$. This implies, thanks to Lemma 1.5, that

$$[w_i, w_j] = [\tilde{w}_i, \tilde{w}_j] \otimes [1]_{k+1} = [g_i^{i'} \tilde{v}_{i'}, g_j^{j'} \tilde{v}_{j'}] \otimes [1]_{k+1} \in \mathcal{F}.$$

We claim now that the \tilde{w}_j generate \mathcal{D} and therefore, when restricted to $S(k)$, generate \mathcal{F} . Let π be the projection $(z^1, \dots, z^n) \mapsto (z^{m+1}, \dots, z^{m+l})$, and let $\Pi = \pi \circ \phi$. We have

$$\Pi_*(\tilde{w}_i) = \Pi_*(\tilde{w}_i) + g_i^t \Pi_* \left(\frac{\partial}{\partial z^t} \right) = \Pi_*(g_i^k \tilde{v}_k) = \Pi_* \left(\frac{\partial}{\partial z^t} \right) = \frac{\partial}{\partial z^t},$$

so the \tilde{w}_i generate \mathcal{D} . Moreover, by naturality of Lie brackets, we have

$$\Pi_*([\tilde{w}_i, \tilde{w}_j]) = [\Pi_*(\tilde{w}_i), \Pi_*(\tilde{w}_j)].$$

The mapping Π_* induces a map $\Pi_{*,k} : \mathcal{T}_M \otimes \mathcal{O}_{S(k)} \rightarrow \mathcal{O}_{S(k)}^l$, given by

$$\Pi_{*,k}(v \otimes [1]_{k+1}) = \Pi_*(\tilde{v}) \otimes [1]_{k+1}.$$

This map is injective when restricted to \mathcal{F} ; since $[w_i, w_j] \in \mathcal{F}$ and $\Pi_{*,k}([w_i, w_j]) = 0$ we have $[w_i, w_j] = 0$. We want now to modify the \tilde{w}_i 's to obtain l independent commuting sections of \mathcal{F} , without changing their equivalence class. Therefore, we look for extensions of the w_i 's which satisfy the thesis of the theorem, proceeding by induction on the number of sections. If $l' = 1$, we can take any extension of w_{m+1} . (Every vector field commutes with itself.) Suppose now that the claim is true for $l' - 1$ sections. Then, by the holomorphic Frobenius theorem there

exists a coordinate chart adapted to S in which $\tilde{w}_{m+1} = \partial/\partial z^{m+1}, \dots, \tilde{w}_{m+l'-1} = \partial/\partial z^{m+l'-1}$. Now, since the w_i are commuting when restricted to $S(k)$, if

$$w_{m+l'} = [g^v]_{k+1} \frac{\partial}{\partial z^v} + [f^i]_{k+1} \frac{\partial}{\partial z^i},$$

we have

$$[0]_{k+1} = \frac{\partial [g^v]_{k+1}}{\partial z^i} \frac{\partial}{\partial z^v} + \frac{\partial [f^j]_{k+1}}{\partial z^i} \frac{\partial}{\partial z^j} = \left[\frac{\partial g^v}{\partial z^i} \right]_{k+1} \frac{\partial}{\partial z^v} + \left[\frac{\partial f^j}{\partial z^i} \right]_{k+1} \frac{\partial}{\partial z^j},$$

where i ranges in $m+1, \dots, m+l'-1$. The last equality tells us that

$$\frac{\partial g^v}{\partial z^i} = z^{r_1} \dots z^{r_{k+1}} h_{r_1, \dots, r_{k+1}, i}^v, \quad \frac{\partial f^j}{\partial z^i} = z^{r_1} \dots z^{r_{k+1}} h_{r_1, \dots, r_{k+1}, i}^j.$$

We have to find $(\tilde{g}^v, \tilde{f}^j)$ -representatives for the classes $[g^v]_{k+1}, [f^j]_{k+1}$ such that

$$0 = \frac{\partial \tilde{g}^v}{\partial z^i} \frac{\partial}{\partial z^v} + \frac{\partial \tilde{f}^j}{\partial z^i} \frac{\partial}{\partial z^j}.$$

We do that for one of the g^v 's; the method applies to all the other coefficients. Now, $\tilde{g}^v = g^v + z^{r_1} \dots z^{r_{k+1}} \tilde{h}_{r_1, \dots, r_{k+1}}^v$, so

$$\begin{aligned} \frac{\partial \tilde{g}^v}{\partial z^i} &= \frac{\partial g^v}{\partial z^i} + z^{r_1} \dots z^{r_{k+1}} \frac{\partial \tilde{h}_{r_1, \dots, r_{k+1}}^v}{\partial z^i} \\ &= z^{r_1} \dots z^{r_{k+1}} h_{r_1, \dots, r_{k+1}, i}^v + z^{r_1} \dots z^{r_{k+1}} \frac{\partial \tilde{h}_{r_1, \dots, r_{k+1}}^v}{\partial z^i}. \end{aligned}$$

Therefore, the problem reduces to finding a primitive $\tilde{h}_{r_1, \dots, r_{k+1}}^v$ for the 1-form

$$\omega := -h_{r_1, \dots, r_{k+1}, i}^v dz^i,$$

where the other coordinates are considered as parameters. If we denote by ∂ the holomorphic differential and suppose, without loss of generality, that U is simply connected and centered at $x \in S$ (i.e., $\phi(x) = 0$) we have, by the conjugate of the $\bar{\partial}$ -lemma, that this primitive exists if and only if ω is ∂ -closed. Therefore we need to check that the mixed partial derivatives coincide:

$$z^{r_1} \dots z^{r_{k+1}} \frac{\partial h_{r_1, \dots, r_{k+1}, i}^v}{\partial z^j} = \frac{\partial^2 g^v}{\partial z^j \partial z^i} = \frac{\partial^2 g^v}{\partial z^i \partial z^j} = z^{r_1} \dots z^{r_{k+1}} \frac{\partial h_{r_1, \dots, r_{k+1}, j}^v}{\partial z^i}.$$

Then, the primitive exists and is defined in U by

$$\tilde{h}_{r_1, \dots, r_{k+1}}^v(z^1, \dots, z^n) = \int_{\gamma} -h_{r_1, \dots, r_{k+1}, i}^v dz^i,$$

where γ is a curve such that $\gamma(1) = (z^1, \dots, z^n)$ and $\gamma(0) = 0$. \square

As a simple consequence of the lemma, we have the Frobenius theorem for k th infinitesimal neighborhoods.

COROLLARY 1.11 (FROBENIUS THEOREM FOR K TH INFINITESIMAL NEIGHBORHOODS)

Suppose that we have a foliation \mathcal{F} of $S(k)$ of rank l . Then there exists an atlas

$\{U_\alpha, \phi_\alpha\}$ adapted to S such that if $U_\alpha \cap U_\beta \cap S \neq \emptyset$, then

$$(3) \quad \left[\frac{\partial z_\alpha^t}{\partial z_\beta^i} \right]_{k+1} = 0,$$

for $t = 1, \dots, m, m+l+1, \dots, n$ and $i = 1, \dots, l$ on $U_\alpha \cap U_\beta$.

Proof

We take an atlas adapted to S and extensions $\tilde{w}_{i,\alpha}$ as given by Lemma 1.10. By the holomorphic Frobenius theorem, there exists a coordinate system (modulo shrinking) on U_α such that

$$\tilde{w}_{m+1,\alpha} = \frac{\partial}{\partial z_\alpha^{m+1}}, \dots, \tilde{w}_{m+l,\alpha} = \frac{\partial}{\partial z_\alpha^{m+l}}.$$

We take such coordinate systems. Since we are dealing with a foliation of $S(k)$, we know that if $U_\alpha \cap U_\beta \cap S \neq \emptyset$ and \mathcal{F} is generated on each $U_\alpha \cap S$ by $w_{1,\alpha}, \dots, w_{l,\alpha}$ we have $w_{i,\alpha} = [(c_{\alpha\beta})^j_i]_{k+1} w_{j,\beta}$. Hence

$$\begin{aligned} \left[\frac{\partial z_\alpha^t}{\partial z_\beta^i} \right]_{k+1} &= \tilde{w}_{i,\beta} \otimes [1]_{k+1}(z_\alpha^t) = w_{i,\beta}(z_\alpha^t) = [c_i^j]_{k+1} w_{j,\alpha}(z_\alpha^t) \\ &= [c_i^j]_{k+1} \tilde{w}_{j,\alpha} \otimes [1]_{k+1}(z_\alpha^t) = [c_i^j]_{k+1} \left[\frac{\partial z_\alpha^t}{\partial z_\alpha^j} \right]_{k+1} = [c_i^j \delta_j^t]_{k+1} = [0]_{k+1}. \quad \square \end{aligned}$$

REMARK 1.12

It is easily seen that the existence of an atlas satisfying (3) implies the existence of a foliation of $\mathcal{T}_{S(k)}$, generated on each chart U_α intersecting S by $\{\partial/\partial z_\alpha^{m+1}, \dots, \partial/\partial z_\alpha^{m+l}\}$.

DEFINITION 1.13

We say that a foliation \mathcal{F} of S extends to the k th infinitesimal neighborhood if there exists an atlas adapted to S and \mathcal{F} such that

$$\left[\frac{\partial z_\beta^t}{\partial z_\alpha^i} \right]_{k+1} = 0,$$

for $t = 1, \dots, m, m+l+1, \dots, n$ and $i = 1, \dots, l$ on $U_\alpha \cap U_\beta$.

In the special case $\mathcal{F} = \mathcal{T}_S$ we say that S has a k th-order extendable tangent bundle.

REMARK 1.14

Let M be a complex manifold, and let \mathcal{F} be a regular foliation of M . Every leaf of \mathcal{F} has a k th-order extendable tangent bundle for every k .

REMARK 1.15

For a submanifold S , having a first-order extendable tangent bundle is a strong topological condition. As a matter of fact, as we see in Section 7 of this paper, this implies the vanishing of many of the characteristic classes of the normal bundle of S .

REMARK 1.16

If a submanifold S has a first-order extendable tangent bundle, it is likely that every foliation on S extends to a foliation of the first infinitesimal neighborhood. A result in this direction can be found in Corollary 4.9.

2. The Atiyah sheaf for the variation action

The Atiyah sheaf is an important geometric object defined in [Ati]. In that paper, it was proved that the existence of a holomorphic connection for the sheaf of sections \mathcal{E} of a holomorphic vector bundle E is equivalent to the splitting of the following sequence:

$$(4) \quad 0 \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{A}_{\mathcal{E}} \rightarrow TM \rightarrow 0,$$

where $\mathcal{A}_{\mathcal{E}}$ is the Atiyah sheaf of \mathcal{E} . In [ABT1] it was proved that the obstruction to the existence of a holomorphic connection for the sheaf of sections \mathcal{E} of a holomorphic vector bundle E along a subsheaf \mathcal{F} is equivalent to the splitting of the following sequence:

$$(5) \quad 0 \longrightarrow \text{Hom}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{A}_{\mathcal{E}, \mathcal{F}} \longrightarrow \mathcal{F} \longrightarrow 0.$$

REMARK 2.1

Please note that in the whole section, \mathcal{F} is a nonsingular foliation of S and \mathcal{T}_M and \mathcal{T}_S are always locally free, due to the fact that M is a manifold and S is a submanifold. Therefore, we do not distinguish between the sheaves and their corresponding vector bundles.

DEFINITION 2.2

Let \mathcal{F} be a foliation of S ; let $\mathcal{T}_{M,S(1)} := \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{S(1)}$ and $\mathcal{T}_{M,S} := \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$; if $\theta_1 : \mathcal{O}_{S(1)} \rightarrow \mathcal{O}_S$ is the canonical projection, we denote by Θ_1 the map $\text{id} \otimes \theta_1 : \mathcal{T}_{M,S(1)} \rightarrow \mathcal{T}_{M,S}$. We see \mathcal{F} as a subsheaf of $\mathcal{T}_{M,S}$; we define the *normal sheaf to the foliation in the ambient tangent sheaf* as the quotient of $\mathcal{T}_{M,S}$ by \mathcal{F} , and we will denote it by $\mathcal{N}_{\mathcal{F},M}$. Let $\mathcal{T}_{M,S(1)}^{\mathcal{F}} := \ker(\text{pr} \circ \Theta_1)$, where pr is the quotient map in the short exact sequence

$$(6) \quad \begin{array}{ccccccc} & & & \mathcal{T}_S & & & \\ & & \nearrow & \downarrow & & & \\ 0 & \longrightarrow & \mathcal{F} & \xrightarrow{i} & \mathcal{T}_{M,S} & \xrightarrow{\text{pr}} & \mathcal{N}_{\mathcal{F},M} \longrightarrow 0. \end{array}$$

REMARK 2.3

In our case, we have to replace \mathcal{E} in (4) with $\mathcal{N}_{\mathcal{F},M}$; the computation of the obstruction to the splitting of this sequence is a straightforward application of the procedure in [Ati], and therefore we omit it. In an atlas adapted to S and \mathcal{F} ,

in Čech–de Rham cohomology the class is represented by the cocycle

$$\left\{ U_{\alpha\beta}, -\frac{\partial z_{\alpha}^{t'}}{\partial z_{\beta}^t} \frac{\partial^2 z_{\beta}^t}{\partial z_{\alpha}^i \partial z_{\alpha}^w} \Big|_S dz_{\alpha}^i \otimes \omega_{\beta}^w \otimes \partial_{t',\beta} \right\},$$

where $\{\partial_{t,\alpha}\}$ is the quotient frame for $\mathcal{N}_{F,M}$ in U_{α} and ω_{α}^t is the dual frame for $\mathcal{N}_{F,M}$ on U_{α} .

As in [ABT2], we will define a more concrete realization of the Atiyah sheaf for the sheaf $\mathcal{N}_{F,M}$. We shall prove that the splitting of the Atiyah sequence for $\mathcal{N}_{F,M}$ is equivalent to the fact that the foliation \mathcal{F} extends to the first infinitesimal neighborhood.

REMARK 2.4

By definition $\Theta_1(\mathcal{T}_{M,S(1)}^{\mathcal{F}})$ is contained in the kernel of pr , so, by exactness of sequence (6), it is contained in the image of \mathcal{F} . Moreover, for each $v \in \mathcal{F}$, at least locally, the element $\tilde{v} \otimes [1]_2$ belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ and is projected by Θ_1 to $i(v)$. So, $\Theta_1(\mathcal{T}_{M,S(1)}^{\mathcal{F}}) = i(\mathcal{F})$.

REMARK 2.5

Suppose that we have a coordinate system adapted to S and \mathcal{F} (see Definition 1.8). Then v belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ if and only if $v = [a^k]_2 \partial / \partial z^k$, with $[a^t]_1 = 0$, where $t = 1, \dots, m, m + l + 1, \dots, n$. Analogously v belongs to $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ if and only if $v = [a^i]_2 \partial / \partial z^i$, where $a^i \in \mathcal{I}_S$ for $i = m + 1, \dots, m + l$.

LEMMA 2.6

Let \mathcal{F} be a foliation of S . Then

- (1) every v in $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ induces a derivation $g \mapsto v(g)$ of $\mathcal{O}_{S(1)}$;
- (2) there exists a natural \mathbb{C} -linear map $\{\cdot, \cdot\} : \mathcal{T}_{M,S(1)}^{\mathcal{F}} \otimes \mathcal{T}_{M,S(1)}^{\mathcal{F}} \rightarrow \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ such that
 - (a) $\{u, v\} = -\{v, u\}$,
 - (b) $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$,
 - (c) $\{gu, v\} = g\{u, v\} - v(g)u$, for all $g \in \mathcal{O}_{S(1)}$,
 - (d) $\Theta_1(\{u, v\}) = [\Theta_1(u), \Theta_1(v)]$.

Proof

(1) Let $(U; z^1, \dots, z^n)$ be a coordinate chart adapted to S and \mathcal{F} . An element $v = [a^k]_2 \frac{\partial}{\partial z^k} \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ if and only if $[a^t]_1 = 0$. Remembering Remark 1.3 we see that v belongs to $\mathcal{T}_{S(1)}$ and Lemma 1.5 gives the assertion.

(2) We define $\{\cdot, \cdot\}$ by setting

$$\{u, v\}(f) = u(v(f)) - v(u(f)),$$

for every $f \in \mathcal{O}_{S(1)}$. Please note that, since u and v belong to $\mathcal{T}_{S(1)}$, this bracket coincides with the bracket defined on $\mathcal{T}_{S(1)}$; the first three properties are proved

exactly as for the usual bracket of vector fields, while the fourth follows from a simple computation in coordinates. Suppose that $(U; z^1, \dots, z^n)$ is a coordinate chart adapted to S and \mathcal{F} , and suppose $u = [a^k]_2 \frac{\partial}{\partial z^k}$, $v = [b^k]_2 \frac{\partial}{\partial z^k}$ with $[a^t]_1 = 0$ and $[b^t]_1 = 0$. First of all we compute the Lie brackets on $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ in coordinates:

$$\begin{aligned} \{u, v\} &= \left[a^h \frac{\partial b^k}{\partial z^h} - b^h \frac{\partial a^k}{\partial z^h} \right]_2 \frac{\partial}{\partial z^k} \\ &= \left[a^t \frac{\partial b^u}{\partial z^t} + a^i \frac{\partial b^u}{\partial z^i} - b^t \frac{\partial a^u}{\partial z^t} - b^i \frac{\partial a^u}{\partial z^i} \right]_2 \frac{\partial}{\partial z^u} \\ &\quad + \left[a^t \frac{\partial b^j}{\partial z^t} - b^t \frac{\partial a^j}{\partial z^t} \right]_2 \frac{\partial}{\partial z^j} + \left[a^i \frac{\partial b^j}{\partial z^i} - b^i \frac{\partial a^j}{\partial z^i} \right]_2 \frac{\partial}{\partial z^j}. \end{aligned}$$

Please note that the coefficients in the first two summands of the last expression all belong to $\mathcal{I}_S/\mathcal{I}_S^2$. Therefore

$$\Theta_1(\{u, v\}) = \left[a^i \frac{\partial b^j}{\partial z^i} - b^i \frac{\partial a^j}{\partial z^i} \right]_1 \frac{\partial}{\partial z^j} = [\Theta_1(u), \Theta_1(v)]. \quad \square$$

REMARK 2.7

In general, given two vector fields u, v in $\mathcal{T}_{M,S(1)}$, we can define a bracket as $[u, v](f) = u(v(f)) - v(u(f))$, for $f \in \mathcal{O}_{S(1)}$. Please note that this bracket is not a well-defined section of $\mathcal{T}_{M,S(1)}$ but only of $\mathcal{T}_{M,S}$. In other words $[u(v(f)) - v(u(f))]_2$ is not well defined, while $[u(v(f)) - v(u(f))]_1$ is.

This shows how, in our treatment, the hypothesis of working with logarithmic vector fields is fundamental; in Lemma 2.6 the bracket operator is well defined since the domain is $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$.

LEMMA 2.8

Let \mathcal{F} be a foliation of S . Then

- (1) $u \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ is such that $\text{pr}([u, s]) = 0$ for all $s \in \mathcal{T}_{M,S(1)}$ if and only if $u \in \mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$;
- (2) if $u \in \mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ and $v \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$, then $\{u, v\} \in \mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$;
- (3) the quotient sheaf

$$\mathcal{A} = \mathcal{T}_{M,S(1)}^{\mathcal{F}} / \mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$$

admits a natural structure of an \mathcal{O}_S locally free sheaf such that the map induced by Θ_1 , whose image lies in \mathcal{F} , is an \mathcal{O}_S -morphism.

Proof

- (1) Writing $u = [a^k]_2 \frac{\partial}{\partial z^k}$, with $[a^t]_1 = 0$, and $s = [b^h]_2 \frac{\partial}{\partial z^h} \in \mathcal{T}_{M,S(1)}$, we have

$$\text{pr}([u, s]) = \left[a^k \frac{\partial b^t}{\partial z^k} - b^k \frac{\partial a^t}{\partial z^k} \right]_1 \frac{\partial}{\partial z^t}.$$

If u belongs to $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ clearly $\text{pr}([u, s]) = 0$.

Conversely, let u be such that $\text{pr}([u, s]) = 0$ for each $s \in \mathcal{T}_{M,S(1)}$. We claim that it belongs to $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$. We know that u belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$, so that $[a^t]_1 = 0$.

Let $s = \partial/\partial z^r$, with $r = 1, \dots, m$. Then $[\partial a^t/\partial z^r]_1 = 0$. Now, we take a representative $h_s z^s$, with $s = 1, \dots, m$, for the class $[a^t]_1$. After computing,

$$0 = \left[\frac{\partial a^t}{\partial z^r} \right]_1 = \left[\frac{\partial h_s}{\partial z^s} z^s + h_s \delta_r^s \right]_1 = [h_s]_1.$$

So, for each s , we have that h_s belongs to \mathcal{I}_S , implying that $[a^t]_2 = 0$. Fix now a j in $m + 1, \dots, n$, and let $s = [z^j]_2 \frac{\partial}{\partial z^1}$. Then

$$0 = - \left[z^j \frac{\partial a^t}{\partial z^1} \right]_1 \frac{\partial}{\partial z^t} + [a^k \delta_k^j]_1 \frac{\partial}{\partial z^1} = [a^j]_1 \frac{\partial}{\partial z^1},$$

where the last equality follows from the preceding step, where we proved that $[a^t]_2 = 0$ and thus that $[\frac{\partial a^t}{\partial z^1}]_1 = 0$. So, $[a^j]_1 = 0$, and u belongs to $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$.

(2) This follows by a direct computation in coordinates.

(3) The sheaf $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ is an $\mathcal{O}_{S(1)}$ -submodule of $\mathcal{T}_{M,S(1)}$ such that $g \cdot v$ belongs to $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ for every $g \in \mathcal{I}_S/\mathcal{I}_S^2$ and $v \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$. Therefore the $\mathcal{O}_{S(1)}$ structure induces a natural \mathcal{O}_S -module structure on \mathcal{A} .

Remember that $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ is generated locally, in an atlas adapted to S by $\partial/\partial z^j$, with $j = m + 1, \dots, m + l$ and by $[z^r]_2 \partial/\partial z^s$, with r and s varying in $1, \dots, m$. Then, the sheaf \mathcal{A} is a locally free \mathcal{O}_S -module freely generated by $\pi\left(\frac{\partial}{\partial z^j}\right)$ and $\pi\left([z^s]_2 \frac{\partial}{\partial z^t}\right)$, where $\pi : \mathcal{T}_{M,S(1)}^{\mathcal{F}} \rightarrow \mathcal{A}$ is the quotient map. Moreover, $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ lies in the kernel of Θ_1 , so Θ_1 factors through a map that we will denote again by $\Theta_1 : \mathcal{A} \rightarrow \mathcal{F}$, which is clearly an \mathcal{O}_S -morphism. □

DEFINITION 2.9

Let \mathcal{F} be a foliation of S . The *Atiyah sheaf of \mathcal{F}* is the locally free \mathcal{O}_S -module

$$\mathcal{A} = \mathcal{T}_{M,S(1)}^{\mathcal{F}}/\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}.$$

THEOREM 2.10

Let \mathcal{F} be a foliation of S . Then there exists a natural exact sequence of locally free \mathcal{O}_S -modules

$$0 \longrightarrow \text{Hom}(\mathcal{N}_S, \mathcal{N}_{\mathcal{F},M}) \longrightarrow \mathcal{A} \xrightarrow{\Theta_1} \mathcal{F} \longrightarrow 0$$

whose splitting is equivalent to the splitting of the sequence (5) taking instead of \mathcal{E} the sheaf $\mathcal{N}_{\mathcal{F},M}$.

Proof

We work in a chart adapted to S and \mathcal{F} . The kernel of Θ_1 is locally freely generated by the images under $\pi : \mathcal{T}_{M,S(1)}^{\mathcal{F}} \rightarrow \mathcal{A}$ of $[z_\alpha^s]_2 \frac{\partial}{\partial z_\alpha^t}$. We would like to understand how the generators behave under change of coordinates to see if $\ker(\Theta_1)$ is isomorphic to any known sheaf of sections of a known vector bundle. We compute the coordinate change maps:

$$(7) \quad \pi\left([z_\alpha^s]_2 \frac{\partial}{\partial z^t}\right) = \pi\left([z_\alpha^s]_2 \left[\frac{\partial z_\beta^k}{\partial z_\alpha^t} \right]_2 \frac{\partial}{\partial z_\beta^k}\right) = \pi\left([z_\alpha^s]_2 \left[\frac{\partial z_\beta^w}{\partial z_\alpha^t} \right]_2 \frac{\partial}{\partial z_\beta^w}\right)$$

$$\begin{aligned}
(8) \quad &= \pi \left(\left[\frac{\partial z_\alpha^s}{\partial z_\beta^{s_1}} \right]_2 [z_\beta^{s_1}]_2 \left[\frac{\partial z_\beta^w}{\partial z_\alpha^t} \right]_2 \frac{\partial}{\partial z_\beta^w} \right) \\
&= \left[\frac{\partial z_\alpha^s}{\partial z_\beta^{s_1}} \frac{\partial z_\beta^w}{\partial z_\alpha^t} \right]_1 \pi \left([z_\beta^{s_1}]_2 \frac{\partial}{\partial z_\beta^w} \right),
\end{aligned}$$

where the last equality in (7) comes from the quotient map and the one in (8) comes from the newly acquired structure of the \mathcal{O}_S -module. As a consequence, the kernel of Θ_1 is isomorphic to $\text{Hom}(\mathcal{N}_S, \mathcal{N}_{\mathcal{F}, M})$. Now, if we define local splittings of the sequence by setting

$$\sigma_\alpha \left(\frac{\partial}{\partial z_\alpha^j} \right) = \pi \left(\frac{\partial}{\partial z_\alpha^j} \right),$$

for $j = m + 1, \dots, m + l$, and extending by \mathcal{O}_S -linearity, we can compute the obstruction to find a splitting of the sequence:

$$\begin{aligned}
(9) \quad &(\sigma_\beta - \sigma_\alpha) \left(\frac{\partial}{\partial z_\beta^j} \right) = \sigma_\beta \left(\frac{\partial}{\partial z_\beta^j} \right) - \sigma_\alpha \left(\left[\frac{\partial z_\alpha^i}{\partial z_\beta^j} \right]_1 \frac{\partial}{\partial z_\alpha^i} \right) \\
&= \sigma_\beta \left(\frac{\partial}{\partial z_\beta^j} \right) - \left[\frac{\partial z_\alpha^i}{\partial z_\beta^j} \right]_1 \sigma_\alpha \left(\frac{\partial}{\partial z_\alpha^i} \right) \\
&= \pi \left(\frac{\partial}{\partial z_\beta^j} \right) - \left[\frac{\partial z_\alpha^i}{\partial z_\beta^j} \right]_1 \pi \left(\frac{\partial}{\partial z_\alpha^i} \right) \\
&= \pi \left(\left[\frac{\partial z_\alpha^t}{\partial z_\beta^j} \right]_2 \frac{\partial}{\partial z_\alpha^t} \right) = \pi \left(\left[\frac{\partial^2 z_\alpha^t}{\partial z_\beta^r \partial z_\beta^j} z_\beta^r \right]_2 \frac{\partial}{\partial z_\alpha^t} \right) \\
&= \left[\frac{\partial^2 z_\alpha^t}{\partial z_\beta^r \partial z_\beta^j} \frac{\partial z_\beta^r}{\partial z_\alpha^s} \right]_1 \pi \left([z_\alpha^s]_2 \frac{\partial}{\partial z_\alpha^t} \right).
\end{aligned}$$

Please note that, since $\partial z_\alpha^t / \partial z_\beta^j$ lies in the ideal \mathcal{I}_S for $t = 1, \dots, m, m + l + 1, \dots, n$, and $j = m + 1, \dots, m + l$, it follows that

$$\frac{\partial^2 z_\alpha^t}{\partial z_\beta^p \partial z_\beta^j} \in \mathcal{I}_S$$

for $t = 1, \dots, m, m + l + 1, \dots, n$, $j = m + 1, \dots, m + l$, and $p = m + 1, \dots, n$. Therefore we have

$$\left[\frac{\partial^2 z_\alpha^t}{\partial z_\beta^w \partial z_\beta^j} \right]_1 = [0]_1$$

for $t, w = 1, \dots, m, m + l + 1, \dots, n$ and $j = m + 1, \dots, m + l$ if and only if

$$\left[\frac{\partial^2 z_\alpha^t}{\partial z_\beta^r \partial z_\beta^j} \right]_1 = [0]_1$$

for $t, w = 1, \dots, m, m + l + 1, \dots, n$, $j = m + 1, \dots, m + l$, and $r = 1, \dots, m$. Hence, class (9) vanishes if and only if (5) splits. \square

It is easily noted that in the case where \mathcal{F} is the tangent sheaf to S the Atiyah sheaf of \mathcal{F} is nothing else but the Atiyah sheaf of S , defined in [ABT2].

DEFINITION 2.11

Let \mathcal{F} be a sheaf of \mathcal{O}_S -modules over a complex manifold S , equipped with an \mathcal{O}_S -morphism $X : \mathcal{F} \rightarrow \mathcal{T}_S$. We say that \mathcal{F} is a *Lie algebroid of anchor X* if there is a \mathbb{C} -bilinear map $\{\cdot, \cdot\} : \mathcal{F} \oplus \mathcal{F} \rightarrow \mathcal{F}$ such that

- (1) $\{v, u\} = -\{u, v\}$;
- (2) $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$;
- (3) $\{g \cdot u, v\} = g \cdot \{u, v\} - X(v)(g) \cdot u$ for all $g \in \mathcal{O}_S$ and $u, v \in \mathcal{F}$.

DEFINITION 2.12

Let \mathcal{E} and \mathcal{F} be locally free sheaves of \mathcal{O}_S -modules over a complex manifold S . Given a section $X \in H^0(S, \mathcal{T}_S \otimes \mathcal{F}^*)$, a *holomorphic X -connection on \mathcal{E}* is a \mathbb{C} -linear map $\tilde{X} : \mathcal{E} \rightarrow \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\tilde{X}(g \cdot s) = X^*(dg) \otimes s + g\tilde{X}(s),$$

for each $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$, where X^* is the dual map of X . The notation $\tilde{X}_v(s)$ is equivalent to $\tilde{X}(s)(v)$.

If \mathcal{F} is a Lie algebroid of anchor X we define the *curvature* of \tilde{X} to be

$$R_{u,v}(s) = \tilde{X}_u \circ \tilde{X}_v(s) - \tilde{X}_v \circ \tilde{X}_u(s) - \tilde{X}_{\{u,v\}}(s).$$

We say that \tilde{X} is *flat* if $R \equiv 0$.

PROPOSITION 2.13

Let \mathcal{F} be a holomorphic foliation of S . Then

- (1) the Atiyah sheaf of \mathcal{F} has a natural structure of a Lie algebroid of anchor Θ_1 such that

$$\Theta_1\{q_1, q_2\} = [\Theta_1(q_1), \Theta_1(q_2)]$$

for all $q_1, q_2 \in \mathcal{A}$;

- (2) there is a natural holomorphic Θ_1 -connection $\tilde{X} : \mathcal{N}_{\mathcal{F},M} \rightarrow \mathcal{A}^* \otimes \mathcal{N}_{\mathcal{F},M}$ on $\mathcal{N}_{\mathcal{F},M}$ given by

$$\tilde{X}_q(s) = \text{pr}([v, \tilde{s}])$$

for all $q \in \mathcal{A}$ and $s \in \mathcal{N}_{\mathcal{F},M}$, where $v \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ and $\tilde{s} \in \mathcal{T}_{M,S(1)}$ are such that $\pi(v) = q$ and $\text{pr} \circ \Theta_1(\tilde{s}) = s$;

- (3) this holomorphic Θ_1 -connection is flat.

Proof

- (1) We set

$$\{q_1, q_2\} = \pi(\{v_1, v_2\}),$$

where $v_i \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$ are such that $q_i = \pi(v_i)$ for $i = 1, 2$. This is well defined: if $q_1 = 0$, then v_1 is in $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$, and then, by Lemma 2.8(2) we have $\{q_1, q_2\} = 0$. The other properties follow directly from Lemma 2.6.

(2) We check that the connection is well defined. Suppose now $q = 0$; this means that $v \in \mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$; then, by Lemma 2.8.1, we have $\text{pr}([v, \tilde{s}]) = 0$, for every $\tilde{s} \in \mathcal{T}_{M,S(1)}$. Now, if $\text{pr} \circ \Theta_1(\tilde{s}) = 0$, we have $\tilde{s} \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$, so $\{v, \tilde{s}\}$ is in $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$, which implies that $\tilde{X}_q(s) = 0$.

We check now that it is a Θ_1 -connection. It is \mathcal{O}_S -linear in the first entry since

$$\tilde{X}_{[f]_1 \cdot q}(s) = \text{pr}([[f]_2 v, \tilde{s}]) = \text{pr}([[f]_1 [v, \tilde{s}] - \tilde{s}([f]_2) \Theta_1(v)]) = [f]_1 \tilde{X}_q(v),$$

where the last equality comes from the fact that v belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$, which is the kernel of $\text{pr} \circ \Theta_1$. We check the Θ_1 -Leibniz rule for the second entry:

$$\begin{aligned} \tilde{X}_q([f]_1 s) &= \text{pr}([v, [f]_2 \tilde{s}]) = \text{pr}([[f]_1 [v, \tilde{s}] + v([f]_2) \cdot \Theta_1(\tilde{s})]) \\ &= [f]_1 \tilde{X}_q(s) + \Theta_1(q)([f]_1) \cdot s, \end{aligned}$$

where the last equality comes from the equality

$$[v([f]_2)]_1 = \Theta_1(v)([f]_1) = \Theta_1(\pi(v))([f]_1),$$

for every $[f]_2 \in \mathcal{O}_{S(1)}$ and for every $v \in \mathcal{T}_{M,S(1)}^{\mathcal{F}}$. Thus, \tilde{X} is a holomorphic Θ_1 -connection.

(3) We compute the curvature:

$$\begin{aligned} R_{q_1, q_2}(s) &= \tilde{X}_{q_1} \circ \tilde{X}_{q_2}(s) - \tilde{X}_{q_2} \circ \tilde{X}_{q_1}(s) - \tilde{X}_{\{q_1, q_2\}}(s) \\ &= \text{pr}([u, \widetilde{\text{pr}([v, \tilde{s}])})] - \text{pr}([v, \widetilde{\text{pr}([u, \tilde{s}])})] - \text{pr}([[u, v], \tilde{s}]). \end{aligned}$$

As we proved before, the connection does not depend on the extension chosen for the second entry, so we can rewrite the expression as

$$\text{pr}([u, [v, \tilde{s}]) - \text{pr}([v, [u, \tilde{s}]) - \text{pr}([[u, v], \tilde{s}]).$$

Computing in coordinates, it follows from the usual Jacobi identity for vector fields that it is identically zero. \square

DEFINITION 2.14

Let \mathcal{F} be a foliation of S . The holomorphic Θ_1 -connection $\tilde{X} : \mathcal{N}_{\mathcal{F},M} \rightarrow \mathcal{A}^* \otimes \mathcal{N}_{\mathcal{F},M}$ just introduced is called the *universal holomorphic connection* on $\mathcal{N}_{\mathcal{F},M}$.

COROLLARY 2.15

Suppose that there exists a foliation \mathcal{F} of the first infinitesimal neighborhood of S . Then, there exists a flat partial holomorphic connection $(\delta, \mathcal{F}|_S)$ on $\mathcal{N}_{\mathcal{F}|_S, M}$ along $\mathcal{F}|_S$.

Proof

We want to define now the splitting map between $\mathcal{F}|_S$ and \mathcal{A} ; in an atlas adapted

to S and \mathcal{F} each of $[1]_2 \otimes \partial/\partial z_\alpha^i$ belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$. Therefore we define $\psi : \mathcal{F}|_S \rightarrow \mathcal{A}$ as

$$\psi : \frac{\partial}{\partial z_\alpha^i} \mapsto \pi\left([1]_2 \otimes \frac{\partial}{\partial z_\alpha^i}\right),$$

for each $i = m + 1, \dots, m + l$, where π is the map from $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ to \mathcal{A} . We compute now the explicit form of the induced partial holomorphic connection. Indeed, let v belong to \mathcal{F} , and let s belong to $\mathcal{N}_{\mathcal{F},M}$; since $\psi(v)$ belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$, if we take a lift \tilde{s} of s to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$, that is, $\text{pr} \circ \Theta_1(\tilde{s}) = s$, we have that the partial holomorphic connection (δ, \mathcal{F}) along \mathcal{F} induced by the universal holomorphic connection for $\mathcal{N}_{\mathcal{F},M}$ is given by:

$$\delta_v(s) = \tilde{X}_{\psi(v)}(s) = \text{pr}([\psi(v), \tilde{s}]).$$

We prove now that this partial holomorphic connection is flat; indeed

$$\delta_u(\delta_v(s)) - \delta_v(\delta_u(s)) - \delta_{[u,v]}(s) = \text{pr}([\tilde{u}, [\tilde{v}, \tilde{s}]] - [\tilde{v}, [\tilde{u}, \tilde{s}]] - [[\tilde{u}, \tilde{v}], \tilde{s}]) = 0,$$

by the Jacobi identity. □

3. Splittings and foliations of the first infinitesimal neighborhood

In this section we deal with a stronger version of splitting (see Definition 3.1). The main idea is that, given a splitting of a submanifold, there exist maps which permit us to “project” vector fields transversal to the first infinitesimal neighborhood into vector fields which are tangential to the first infinitesimal neighborhood.

Proposition 2.7 of [ABT2], which follows from [Ei, Proposition 16.2], proves that the splitting of the conormal sequence

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{d_2} \Omega_M \otimes \mathcal{O}_S \xrightarrow{p} \Omega_S \longrightarrow 0$$

is equivalent to the splitting of the following short exact sequences:

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{i_1} \mathcal{O}_{S(1)} \xrightarrow{\theta_1} \mathcal{O}_S \longrightarrow 0.$$

If one of these sequences splits, then also the following sequence splits:

$$0 \longrightarrow \mathcal{T}_S \xrightarrow{\iota} \mathcal{T}_{M,S} \xrightarrow{p_2} \mathcal{N}_S \longrightarrow 0.$$

DEFINITION 3.1

We say that S splits in M if there exists a morphism of sheaves $\sigma : \Omega_S \rightarrow \Omega_{M,S}$ such that $p \circ \sigma = \text{id}$ where $p : \Omega_{M,S} \rightarrow \Omega_S$ is the canonical projection.

REMARK 3.2

In [ABT2] it is proved that a submanifold splits if and only if there exists an atlas adapted to S such that

$$\left[\frac{\partial z_\beta^p}{\partial z_\alpha^r} \right]_1 = 0.$$

A natural generalization of the concept of splitting is the notion of k -splitting, developed in [ABT2] and [ABT3]. We will use extensively the notion of 2-splitting.

DEFINITION 3.3

We say that S k -splits into M if and only if there is an infinitesimal retraction of $S(k)$ onto S , that is, if there is a k th-order lifting $\rho : \mathcal{O}_S \rightarrow \mathcal{O}_M/\mathcal{I}_S^{k+1}$ or, in still other words, if the exact sequence

$$(10) \quad 0 \rightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \rightarrow \mathcal{O}_M/\mathcal{I}_S^{k+1} \rightarrow \mathcal{O}_M/\mathcal{I}_S \rightarrow 0$$

splits as a sequence of sheaves of rings.

REMARK 3.4

Please note that, in the case where the sequence above splits, the map $\tilde{\rho} : \mathcal{O}_M/\mathcal{I}_S^{k+1} \rightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1}$ is a θ_k -derivation; that is,

$$\tilde{\rho}([fg]_{k+1}) = \theta_k([f]_{k+1})\tilde{\rho}([g]_{k+1}) + \theta_k([g]_{k+1})\tilde{\rho}([f]_{k+1}).$$

The sheaf $\mathcal{I}_S/\mathcal{I}_S^{k+1}$ has a natural structure of an $\mathcal{O}_{S(1)}$ -module: the multiplication given by $[f]_k[h]_{k+1} = [fh]_{k+1}$ is well defined; indeed, let $[\tilde{f}_1]_{k+1}$ and $[\tilde{f}_2]_{k+1}$ be two representatives of $[f]_k$. Then $[\tilde{f}_2 - \tilde{f}_1]_{k+1}$ belongs to $\mathcal{I}_S^k/\mathcal{I}_S^{k+1}$, and therefore $[\tilde{f}_1 h]_{k+1} - [\tilde{f}_2 h]_{k+1} = [0]_{k+1}$ since h belongs to $\mathcal{I}_S/\mathcal{I}_S^{k+1}$.

REMARK 3.5

Theorem 2.1 of [ABT3] proves that S is k -splitting if and only if there exists a k -splitting atlas, that is, an atlas $\{U_\alpha, z_\alpha\}$ adapted to a complex submanifold S such that

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^k$$

for all $r = 1, \dots, m, p = m + 1, \dots, n$ and for each pair of indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$.

DEFINITION 3.6

If \mathcal{F} is foliation of M of rank l strictly smaller than the dimension of S and if we denote by σ^* the map from $\mathcal{T}_{M,S}$ to \mathcal{T}_S given in [ABT2, Proposition 2.7], we shall denote by \mathcal{F}^σ the coherent sheaf of \mathcal{O}_S -modules given by

$$\mathcal{F}^\sigma := \sigma^*(\mathcal{F}|_S).$$

We shall say that σ^* is \mathcal{F} -faithful outside an analytic subset $\Sigma \subset S$ if \mathcal{F}^σ is a regular foliation of S of rank l on $S \setminus \Sigma$. If $\Sigma = \emptyset$ we shall simply say that σ^* is \mathcal{F} -faithful.

We refer to [ABT2] for a treatment of \mathcal{F} -faithfulness in the case of splittings. Assume that σ^* is \mathcal{F} -faithful; an interesting question is whether there exists an analogue of σ^* from $\mathcal{T}_{M,S(1)}$ to $\mathcal{T}_{S(1)}$, which restricted to $\mathcal{T}_{M,S}$ coincides with σ^* ;

this would permit us to project a transversal foliation to a foliation of the first infinitesimal neighborhood.

First of all, we can suppose that we are working on a splitting submanifold S .

DEFINITION 3.7

We will call the sheaf $\mathcal{T}_{M,S(k)} := \mathcal{T}_M \otimes \mathcal{O}_{S(k)}$ the *restriction of the ambient tangent sheaf to the k th infinitesimal neighborhood*.

We look for a splitting of the following sequence:

$$(11) \quad 0 \rightarrow \mathcal{T}_{S(1)} \rightarrow \mathcal{T}_{M,S(1)} \rightarrow \mathcal{N}_{S(1)} \rightarrow 0,$$

where $\mathcal{N}_{S(1)}$ is the quotient of the two modules.

REMARK 3.8

Let $(U_\alpha, z_\alpha^1, \dots, z_\alpha^n)$ be a coordinate system adapted to S . Please remember Remark 1.3; since S is a submanifold the ideal of S is generated by $z_\alpha^1, \dots, z_\alpha^r$, and we have that $\mathcal{T}_{S(1)}$ is generated in U_α by

$$[z_\alpha^r]_2 \frac{\partial}{\partial z_\alpha^s}, \frac{\partial}{\partial z_\alpha^{m+1}}, \dots, \frac{\partial}{\partial z_\alpha^n},$$

for r, s varying in $1, \dots, m$, while $\mathcal{T}_{M,S(1)}$ is generated on U_α by

$$\frac{\partial}{\partial z_\alpha^1}, \dots, \frac{\partial}{\partial z_\alpha^n}.$$

Let $\partial_{r,\alpha}$ be the image of $\partial/\partial z_\alpha^r$ in $\mathcal{N}_{S(1)}$, and let ω_α^r be its dual element. Now let

$$v = [f_\alpha^k]_2 \frac{\partial}{\partial z_\alpha^k}$$

be a section of $\mathcal{T}_{M,S(1)}$; we can see that the image of v in $\mathcal{N}_{S(1)}$ is nothing else but $[f_\alpha^r]_1 \partial_{r,\alpha}$. We denote by $[v]$ the equivalence class of v in $\mathcal{N}_{S(1)}$; please note that, given a function $[g]_2$ in $\mathcal{O}_{S(1)}$, the $\mathcal{O}_{S(1)}$ -module structure is given by

$$[g]_2 \cdot [v] = [\theta_1([g]_2) \cdot v].$$

We compute now the transition functions of $\mathcal{N}_{S(1)}$; if we are in an atlas adapted to S we have $z_\alpha^s = h_{\alpha\beta,r}^s z_\beta^{r'}$. We have

$$\begin{aligned} \partial_{r,\alpha} &= \left[\frac{\partial}{\partial z_\alpha^r} \right] = \left[\frac{\partial z_\beta^k}{\partial z_\alpha^r} \frac{\partial}{\partial z_\beta^k} \right] = \left[\frac{\partial z_\beta^s}{\partial z_\alpha^r} \frac{\partial}{\partial z_\beta^s} \right] = \left[\frac{\partial(h_{\alpha\beta,r}^s z_\beta^{r'})}{\partial z_\alpha^r} \frac{\partial}{\partial z_\beta^s} \right] \\ &= \left[\frac{\partial(h_{\alpha\beta,r}^s)}{\partial z_\alpha^r} z_\beta^{r'} \frac{\partial}{\partial z_\beta^s} \right] + \left[h_{\alpha\beta,r}^s \delta_r^{r'} \frac{\partial}{\partial z_\beta^s} \right] = [h_{\alpha\beta,r}^s]_2 \partial_{s,\beta}, \end{aligned}$$

where the last equality comes from the equivalence relations that define $\mathcal{N}_{S(1)}$.

REMARK 3.9

Please note that the transition functions for $(\mathcal{N}_{S(1)})^*$ as an $\mathcal{O}_{S(1)}$ -module are

given by

$$\omega_\beta^s = [h_{\alpha\beta,r}^s]_2 \omega_\alpha^r.$$

Please note that $(\mathcal{N}_{S(1)})^*$ is isomorphic to $\mathcal{I}_S/\mathcal{I}_S^2$ with the structure of an $\mathcal{O}_{S(1)}$ -module given by the projection $\theta_1 : \mathcal{O}_{S(1)} \rightarrow \mathcal{O}_S$.

LEMMA 3.10

Let M be an n -dimensional complex manifold, and let S be a submanifold of codimension r . Then sequence (11) splits if S is 2-splitting, that is, there exists an atlas adapted to S such that

$$(12) \quad \left[\frac{\partial z_\beta^p}{\partial z_\alpha^r} \right]_2 \equiv [0]_2,$$

for $p = m + 1, \dots, n$ and $r = 1, \dots, m$.

Proof

We have to compute the image in $H^1(S, \text{Hom}(\mathcal{N}_{S(1)}, \mathcal{T}_{S(1)}))$ through the coboundary operator of cochain $\{U_\alpha \cap S, \omega_\alpha^r \otimes \partial_{r,\alpha}\}$ representing the identity in $H^0(\mathcal{U}, \text{Hom}(\mathcal{N}_{S(1)}, \mathcal{N}_{S(1)}))$. We compute then

$$(13) \quad \begin{aligned} \delta(U_\alpha, \omega_\alpha^r \otimes \partial_{r,\alpha}) &= \omega_\beta^r \otimes \frac{\partial}{\partial z_\beta^r} - \omega_\alpha^s \otimes \frac{\partial}{\partial z_\alpha^s} \\ &= \omega_\beta^r \otimes \frac{\partial}{\partial z_\beta^r} - \left[\frac{\partial z_\alpha^s}{\partial z_\beta^{r'}} \frac{\partial z_\beta^k}{\partial z_\alpha^s} \right]_2 \omega_\beta^{r'} \otimes \frac{\partial}{\partial z_\beta^k} \\ &= \omega_\beta^r \otimes \frac{\partial}{\partial z_\beta^r} - \left[\frac{\partial z_\alpha^s}{\partial z_\beta^{r'}} \frac{\partial z_\beta^r}{\partial z_\alpha^s} \right]_2 \omega_\beta^{r'} \otimes \frac{\partial}{\partial z_\beta^r} \\ &\quad - \left[\frac{\partial z_\beta^p}{\partial z_\alpha^s} \frac{\partial z_\alpha^s}{\partial z_\beta^{r'}} \right]_2 \omega_\beta^{r'} \otimes \frac{\partial}{\partial z_\beta^p} \\ &= - \left[\frac{\partial z_\beta^p}{\partial z_\alpha^s} \frac{\partial z_\alpha^s}{\partial z_\beta^{r'}} \right]_2 \omega_\beta^{r'} \otimes \frac{\partial}{\partial z_\beta^p}. \end{aligned}$$

This class is clearly zero if we are using a 2-splitting atlas. □

REMARK 3.11

In the last equality of the computation above there is marginal subtle point. If S is 2-splitting, then it is splitting. We saw above that this implies that in an atlas adapted to S and to the splitting

$$\frac{\partial z_\alpha^p}{\partial z_\beta^r} \in \mathcal{I}_S, \quad \frac{\partial z_\alpha^r}{\partial z_\beta^p} \in \mathcal{I}_S.$$

We know also that

$$\frac{\partial z_\alpha^k}{\partial z_\beta^r} \frac{\partial z_\beta^s}{\partial z_\alpha^k} = \delta_r^s.$$

Restricting ourselves to the first infinitesimal neighborhood we have

$$[\delta_r^s]_2 = \left[\frac{\partial z_\alpha^k}{\partial z_\beta^r} \frac{\partial z_\beta^s}{\partial z_\alpha^k} \right]_2 = \left[\frac{\partial z_\alpha^{r'}}{\partial z_\beta^r} \frac{\partial z_\beta^s}{\partial z_\alpha^{r'}} \right]_2 + \left[\frac{\partial z_\alpha^p}{\partial z_\beta^r} \frac{\partial z_\beta^s}{\partial z_\alpha^p} \right]_2 = \left[\frac{\partial z_\alpha^{r'}}{\partial z_\beta^r} \frac{\partial z_\beta^s}{\partial z_\alpha^{r'}} \right]_2,$$

using the splitting hypothesis.

REMARK 3.12

Looking at how we have constructed the splitting in the former lemma, if $\tilde{\rho}$ is the θ_1 -derivation associated to the splitting of S , we have that the splitting morphism $\sigma^* : \mathcal{T}_{M,S(1)} \rightarrow \mathcal{T}_{S(1)}$ is given in an atlas adapted to the two splitting by

$$f^k \frac{\partial}{\partial z^k} \mapsto \tilde{\rho}(f^r) \frac{\partial}{\partial z^r} + f^p \frac{\partial}{\partial z^p}.$$

Now the natural question is under which conditions the splitting of sequence (11) is equivalent to the existence of a 2-splitting atlas. It seems like the splitting of this sequence is not enough. Indeed, if we try to follow the usual approach in proving the argument, as in [ABT3, Theorem 2.1], we have some problems. The first thing we can remark is that the dual of $\mathcal{T}_{M,S(1)}$ is nothing else but $\Omega_M \otimes \mathcal{O}_{S(1)}$. Now, a splitting of (11) implies that there exists a map γ from $\Omega_M \otimes \mathcal{O}_{S(1)}$ to $(\mathcal{N}_{S(1)})^*$ and, since we remarked that $(\mathcal{N}_{S(1)})^*$ is isomorphic to $\mathcal{I}_S/\mathcal{I}_S^2$ as an $\mathcal{O}_{S(1)}$ -module, through a map

$$\tau : (\mathcal{N}_{S(1)})^* \rightarrow \mathcal{I}_S/\mathcal{I}_S^2.$$

This gives rise to a splitting of the map

$$d_2 : \mathcal{I}_S/\mathcal{I}_S^2 \rightarrow \Omega_M \otimes \mathcal{O}_{S(1)}$$

which sends an element $[f]_2$ of $\mathcal{I}_S/\mathcal{I}_S^2$ into $df \otimes [1]_2$. Now, there exists a well-defined map d_3 from $\mathcal{O}_M/\mathcal{I}_S^3$ to $\Omega_M \otimes \mathcal{O}_{S(1)}$, which sends a class $[f]_3$ to $d\tilde{f} \otimes [1]_2$. The big problem is that, even if we suppose S to be comfortably embedded (see [ABT3]), that is, the sequence

$$0 \longrightarrow \mathcal{I}_S^2/\mathcal{I}_S^3 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^3 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow 0$$

splits, we have that the splitting of (11) only gives us a map between $\mathcal{O}_{S(1)}$ and the image through the splitting $\nu : \mathcal{I}_S/\mathcal{I}_S^3 \rightarrow \mathcal{I}_S/\mathcal{I}_S^2$ of $\mathcal{I}_S/\mathcal{I}_S^3$ in $\mathcal{I}_S/\mathcal{I}_S^3$ and this map is not surjective. Therefore it is not a $\theta_{2,1}$ -derivation splitting the short exact sequence of morphisms of rings

$$0 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^3 \longrightarrow \mathcal{O}_{S(1)} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

REMARK 3.13

To solve this problem we could find under which conditions there exists a splitting of the map $\tilde{d}_3 : \mathcal{I}_S^2/\mathcal{I}_S^3 \rightarrow \Omega_{M,S(1)}$. Using such a splitting and the comfortable embedding we could find a $\theta_{2,1}$ -derivation splitting $\iota : \mathcal{I}_S/\mathcal{I}_S^3 \rightarrow \mathcal{O}_{S(1)}$.

We define now a notion parallel to the one in Definition 3.6.

DEFINITION 3.14

If \mathcal{F} is foliation of M of rank l strictly smaller than dimension S and if we denote by σ_2^* the map from $\mathcal{T}_{M,S(1)}$ to $\mathcal{T}_{S(1)}$ splitting sequence (11), we shall denote by \mathcal{F}^{σ_2} the coherent sheaf of $\mathcal{O}_S(1)$ -modules given by

$$\mathcal{F}^{\sigma_2} := \sigma_2^*(\mathcal{F}|_{S(1)}).$$

We shall say that σ_2^* is *first-order \mathcal{F} -faithful outside an analytic subset $\Sigma \subset S$* if \mathcal{F}^{σ_2} is a regular foliation of $S(1)$ of rank l on $S \setminus \Sigma$. If $\Sigma = \emptyset$ we shall simply say that σ_2^* is *first-order \mathcal{F} -faithful*.

Speaking of first-order \mathcal{F} -faithfulness we have a simple results which gives us some insight.

LEMMA 3.15

Let S be 2-splitting in M , and let \mathcal{F} be a one-dimensional holomorphic foliation on M . Let $\sigma_2^* : \mathcal{T}_{M,S(1)} \rightarrow \mathcal{T}_{S(1)}$ be the splitting morphism. If $\sigma_2^*|_S$ is \mathcal{F} -faithful outside an analytic subset Σ , then σ_2^* is first-order \mathcal{F} -faithful outside Σ .

Proof

We check that \mathcal{F}^{σ_2} satisfies the requests of Definition 1.6. By hypothesis $\mathcal{F}^{\sigma_2}|_S$ is a foliation of S . Since the rank of \mathcal{F}^{σ_2} is 1 it is an involutive subbundle of $\mathcal{T}_{S(1)}$; moreover, for each point $x \in S \setminus \Sigma$ we can find a generator v of $\mathcal{F}_x^{\sigma_2}$ such that $v|_S$ is nonzero. Therefore, $\mathcal{T}_{S(1),x}/\mathcal{F}_x^{\sigma_2}$ is $\mathcal{O}_{S(1)}$ -free. \square

Directly from this last lemma and [ABT2, Lemma 7.6] we have the following.

COROLLARY 3.16

Let S be the splitting in M , and let \mathcal{F} be a holomorphic foliation on M of dimension equal to 1 or to the dimension of S . If there exists $x_0 \in S \setminus \text{Sing}(F)$ such that \mathcal{F} is tangent to S at x_0 , that is, $(\mathcal{F}|_S)x_0 \subset \mathcal{T}_{S,x_0}$, then any 2-splitting morphism is first-order \mathcal{F} -faithful outside a suitable analytic subset of S .

4. Extension of foliations and embedding in the normal bundle

DEFINITION 4.1

Let $S(1)$ be the first infinitesimal neighborhood of S , and let $S_N(1)$ be the first infinitesimal neighborhood of the embedding of S as the zero section of its normal bundle in M . We denote by \mathcal{O}_{N_S} the structure sheaf of the normal bundle of S and by \mathcal{I}_{S,N_S} the ideal sheaf of S in N_S . We say that $S_N(1)$ is *isomorphic* to $S(1)$ if there exists an isomorphism $\phi : \mathcal{O}_{N_S}/\mathcal{I}_{S,N_S}^2 \rightarrow \mathcal{O}_M/\mathcal{I}_S^2$ such that $\theta_1 \circ \phi = \theta_1^N$, where $\theta_1 : \mathcal{O}_M/\mathcal{I}_S^2 \rightarrow \mathcal{O}_S$ and $\theta_1^N : \mathcal{O}_{N_S}/\mathcal{I}_{S,N_S}^2 \rightarrow \mathcal{O}_S$ are the canonical projections.

Proposition 1.3 in [ABT2] tells us that, for a splitting submanifold S in M , the first infinitesimal neighborhood in the normal bundle is isomorphic to its infinitesimal neighborhood in M .

REMARK 4.2

In general, given a vector bundle E over a submanifold S , we have that $TE|_S$ is canonically isomorphic to $TS \oplus E$. When E is N_S this implies that the projection on the second summand of $TN_S|_S = TS \oplus N_S$ gives rise to an isomorphism of N_S and N_{0_S} , that is, the normal bundle of S as the zero section of N_S . Therefore we have an isomorphism between $\mathcal{I}_S/\mathcal{I}_S^2$ and $\mathcal{I}_{S,N_S}/\mathcal{I}_{S,N_S}^2$.

Let \mathcal{F} be a foliation of S . Thanks to the holomorphic Frobenius theorem, we know that there exists an atlas $\{(U_\alpha; z_\alpha^1, \dots, z_\alpha^n)\}$ adapted to S and \mathcal{F} . In such an atlas we know that $\mathcal{F} = \ker(dz_\alpha^{m+l+1}|_S, \dots, dz_\alpha^n|_S)$. An equivalent formulation of the Frobenius theorem states that a submodule of $\Omega^1(S)$ is integrable if and only if each stalk is generated by exact forms. We denote by $\pi : N_S \rightarrow S$ the normal bundle of S . The map π is holomorphic; therefore $\pi^*(dz_\alpha^k|_S)$ is a well-defined local holomorphic 1-form on $\pi^{-1}(U_\alpha) \subset N_S$. Moreover, since $\{dz_\alpha^{m+l+1}|_S, \dots, dz_\alpha^n|_S\}$ is an integrable system of 1-forms, so is $\{\pi^*(dz_\alpha^{m+l+1}|_S), \dots, \pi^*(dz_\alpha^n|_S)\}$. Then $\{\pi^*(dz_\alpha^{m+l+1}|_S), \dots, \pi^*(dz_\alpha^n|_S)\}$ defines a foliation $\tilde{\mathcal{F}}$ of N_S , whose leaves are the preimages of the leaves of \mathcal{F} through π . Since S is regular, TM is trivialized on each coordinate neighborhood and so is N_S . In the following we use the atlas $\{(\pi^{-1}(U_\alpha), v_\alpha^1, \dots, v_\alpha^m, z_\alpha^{m+1}, \dots, z_\alpha^n)\}$ of N_S given by the trivializations of the normal bundle, where v_α^r are the coordinates in the fiber; then $\tilde{\mathcal{F}}$ is generated on $\pi^{-1}(U_\alpha)$ by

$$\frac{\partial}{\partial v_\alpha^1}, \dots, \frac{\partial}{\partial v_\alpha^m}, \frac{\partial}{\partial z_\alpha^{m+1}}|_S, \dots, \frac{\partial}{\partial z_\alpha^{m+l}}|_S.$$

The fibers of π are the leaves of a holomorphic foliation of N_S , called the *vertical foliation*, which we denote by \mathcal{V} . On $\pi^{-1}(U_\alpha)$ it is generated by

$$\frac{\partial}{\partial v_\alpha^1}, \dots, \frac{\partial}{\partial v_\alpha^m}.$$

We study now the splitting of the following sequence, when restricted to the first infinitesimal neighborhood of the embedding of S as the zero section of N_S :

$$(14) \quad 0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{\text{pr}} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0.$$

A result of Grothendieck [Gro] tells us that the splitting of the sequence is equivalent to the vanishing of a cohomology class in $H^1(M, \text{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \mathcal{V}))$. The splitting of this sequence is equivalent to the fact that there exists an isomorphism $\tilde{\mathcal{F}} \simeq \mathcal{V} \oplus \tilde{\mathcal{F}}/\mathcal{V}$ compatible with the projection pr and the map ι . Even though this result was already used implicitly in Section 2 we sketch a proof to show how it can be used operatively in our work. Indeed, let ω be the cohomology class associated to the splitting of a short exact sequence of sheaves

$$(15) \quad 0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0;$$

this obstruction is the image of the identity homomorphism in $H^0(M, \text{Hom}(\mathcal{G}, \mathcal{G}))$ into $H^1(M, \text{Hom}(\mathcal{G}, \mathcal{E}))$. By the long exact sequence theorem for Čech cohomology we compute ω in the following way: let $\{U_\alpha, \text{Id}\}$ be the class representing the

identity in $H^0(M, \text{Hom}(\mathcal{G}, \mathcal{G}))$; we take a lift (U_α, τ_α) in $C^0(\mathcal{U}, \text{Hom}(\mathcal{G}, \mathcal{F}))$ and take its Čech coboundary, $\{U_{\alpha\beta}, \tau_\beta - \tau_\alpha\}$. Clearly, $\text{pr} \circ \tau_\beta - \text{pr} \circ \tau_\alpha = 0$, so this is a well-defined element of $C^1(\mathcal{U}, \text{Hom}(\mathcal{G}, \mathcal{E}))$. By diagram chasing, it is shown that this is a Čech cocycle which represents ω . Suppose now that ω is zero in cohomology; this means there exists a cochain $\{U_\alpha, \sigma_\alpha\}$ in $C^0(\mathcal{U}, \text{Hom}(\mathcal{G}, \mathcal{E}))$ whose coboundary is ω , that is, $\sigma_\beta - \sigma_\alpha = \tau_\beta - \tau_\alpha$. We define now a Čech cochain in $C^0(\mathcal{U}, \text{Hom}(\mathcal{E} \oplus \mathcal{G}, \mathcal{F}))$ as $\{U_\alpha, \theta_\alpha\}$ where θ_α is defined on each U_α as

$$\theta_\alpha : (v, w) \mapsto (\iota(v - \sigma_\alpha(w)) + \tau_\alpha(w)).$$

We compute now $\delta\{U_\alpha, \theta_\alpha\}$; on each $U_{\alpha\beta}$:

$$\begin{aligned} & \iota(v - \sigma_\beta(w)) + \tau_\beta(w) - \iota(v - \sigma_\alpha(w)) + \tau_\alpha(w) \\ &= \iota(\sigma_\alpha(w) - \sigma_\beta(w)) + \tau_\beta(w) - \tau_\alpha(w) = 0. \end{aligned}$$

So, we have a global isomorphism of sheaves between $\mathcal{E} \oplus \mathcal{G}$ and \mathcal{F} satisfying our requests.

REMARK 4.3

Please note that $\tilde{\mathcal{F}}/\mathcal{V}$ when restricted to S is nothing else but the foliation \mathcal{F} . This follows directly from our construction of $\tilde{\mathcal{F}}$ as the pullback foliation defined by the integrable system $\{\pi^*(dz_\alpha^{m+l+1}|_S), \dots, \pi^*(dz_\alpha^n|_S)\}$.

LEMMA 4.4

Let S be a splitting in M . If there exists a foliation of the first infinitesimal neighborhood of the embedding of S as the zero section of its normal bundle, then there exists a foliation of the first infinitesimal neighborhood of S embedded in M .

Proof

If there exists a foliation of the first infinitesimal neighborhood of the embedding of S as the zero section of its normal bundle, we can find an atlas of N_S given by $\{V_\alpha, u_\alpha^1, \dots, u_\alpha^m, z_\alpha^{m+1}, \dots, z_\alpha^n\}$ such that, if $V_\alpha \cap V_\beta \cap S \neq \emptyset$ we have

$$\left[\frac{\partial u_\alpha^r}{\partial z_\beta^i} \right]_2 = [0]_2, \quad \frac{\partial z_\alpha^{t'}}{\partial z_\beta^i} = 0,$$

where $r = 1, \dots, m$ and $t' = m + l + 1, \dots, n$.

We use the isomorphism $\phi : \mathcal{O}_{N_S}/\mathcal{I}_{S, N_S}^2 \rightarrow \mathcal{O}_M/\mathcal{I}_S^2$, taking the images

$$[\tilde{z}_\alpha^1]_2 = \phi([u_\alpha^1]_2), \dots, [\tilde{z}_\alpha^r]_2 = \phi([u_\alpha^r]_2), \quad [\tilde{z}_\alpha^{m+1}]_2 = \phi([z_\alpha^{m+1}]_2), \dots, [\tilde{z}_\alpha^n]_2 = \phi([z_\alpha^n]_2);$$

there exist open sets $U_\alpha \supset \pi(V_\alpha)$ (modulo shrinking) where we can choose representatives of these classes such that $(U_\alpha, \tilde{z}_\alpha^1, \dots, \tilde{z}_\alpha^n)$ is a coordinate system adapted to S and \mathcal{F} . If $U_\alpha \cap U_\beta \cap S \neq \emptyset$ we can check that, since $\partial/\partial \tilde{z}_\beta^{m+1}, \dots, \partial/\partial \tilde{z}_\alpha^{m+l}$ are logarithmic

$$\frac{\partial [\tilde{z}_\beta^r]_2}{\partial \tilde{z}_\alpha^i} = \left[\frac{\partial \tilde{z}_\beta^r}{\partial \tilde{z}_\alpha^i} \right]_2 = \left[\frac{\partial u_\beta^r}{\partial z_\alpha^i} \right]_2 = [0]_2,$$

for $r = 1, \dots, m$ and $i = m + 1, \dots, m + l + 1$. Following the same line of thought

$$\frac{\partial[z_\beta^{t'}]_2}{\partial z_\alpha^i} = \left[\frac{\partial z_\beta^{t'}}{\partial z_\alpha^i} \right]_2 = \left[\frac{\partial z_\beta^{t'}}{\partial z_\alpha^i} \right]_2 = [0]_2,$$

for $t' = m + l + 1, \dots, n$ and $i = m + 1, \dots, m + l + 1$. □

So, the problem of extending a foliation outside a submanifold boils down in the splitting case to understanding when (14) splits, and the image through the splitting of \mathcal{F}/\mathcal{V} is involutive. We start by finding a sufficient condition for this to happen.

PROPOSITION 4.5

Let M be a complex manifold of dimension n , and let S be a splitting codimension m submanifold. Let \mathcal{F} be a foliation of S , and let $\pi : N_S \rightarrow M$ be the normal bundle of S in M . Let $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$, and let \mathcal{V} be the vertical foliation given by $\ker d\pi$. The sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{\text{pr}} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0$$

splits if there exists an atlas adapted to \mathcal{F} and S such that

$$\frac{\partial^2 z_\alpha^r}{\partial z_\beta^i \partial z_\beta^s} \in \mathcal{I}_S,$$

for all $r, s = 1, \dots, m$ and $i = m + 1, \dots, m + l$.

Proof

We compute the obstruction to the splitting of the sequence, following [Ati] and [Gro]: we apply the functor $\text{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \cdot)$ to sequence (14) and compute the image of the identity through the coboundary map

$$\delta : H^0(S, \text{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \tilde{\mathcal{F}}/\mathcal{V})) \rightarrow H^1(S, \text{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \mathcal{V})).$$

We fix an atlas $\{U_\alpha, z_\alpha\}$ adapted to S and \mathcal{F} , and we denote the quotient frame for $\tilde{\mathcal{F}}/\mathcal{V}$ by $\{\partial_{m+1,\alpha}, \dots, \partial_{m+l,\alpha}\}$ (i.e., $\partial_{m+1,\alpha}$ is the equivalence class of $\partial/\partial z_\alpha^{m+1}|_S$) and by $\{\omega_\alpha^{m+1}, \dots, \omega_\alpha^{m+l}\}$ its dual frame. The cocycle representing the identity in $H^0(S, \text{Hom}(\tilde{\mathcal{F}}/\mathcal{V}, \tilde{\mathcal{F}}/\mathcal{V}))$ is then represented as $\{U_\alpha, \omega_\alpha^j \otimes \partial_{j,\alpha}\}$; the obstruction to the splitting of the sequence is then

$$\begin{aligned} \delta\{\omega_\alpha^j \otimes \partial_{j,\alpha}\} &= \omega_\beta^j \otimes \frac{\partial}{\partial z_\alpha^j} - \omega_\alpha^j \otimes \frac{\partial}{\partial z_\alpha^j} \\ (16) \quad &= \omega_\beta^j \otimes \frac{\partial}{\partial z_\alpha^j} - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial z_\beta^{j'}}{\partial z_\alpha^j} \right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial z_\beta^{j'}} - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial v_\beta^r}{\partial z_\alpha^j} \right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial v_\beta^r} \\ &= - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial v_\beta^r}{\partial z_\alpha^j} \right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial v_\beta^r}. \end{aligned}$$

The vanishing of (16) is a sufficient condition for the splitting of the sequence; this class vanishes if $\partial v_\beta^r / \partial z_\alpha^j$ belong to $\mathcal{I}_{N_S}^2$. Moreover, the coordinate-change maps of N_S have a peculiar structure:

$$v_\beta^r = v_\alpha^s \frac{\partial z_\alpha^r}{\partial z_\beta^s}.$$

Therefore

$$\begin{aligned} - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial v_\beta^r}{\partial z_\alpha^j} \right]_2 &= - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial}{\partial z_\alpha^j} \left(\frac{\partial z_\alpha^r}{\partial z_\beta^s} \right) \right]_2 \\ &= - \left[v_\alpha^s \frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial^2 z_\alpha^r}{\partial z_\alpha^j \partial z_\beta^s} \right]_2 = - \left[v_\alpha^s \frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial^2 z_\alpha^r}{\partial z_\alpha^s \partial z_\beta^j} \right]_2; \end{aligned}$$

using the isomorphism between $\mathcal{I}_S / \mathcal{I}_S^2$ and $\mathcal{I}_{S, N_S} / \mathcal{I}_{S, N_S}^2$ we see that the last expression vanishes if $\partial z_\alpha^s / \partial z_\beta^j \in \mathcal{I}_S^2$. \square

REMARK 4.6

Since we are working in an atlas of N_S adapted to S and \mathcal{F} we have

$$\frac{\partial z_\alpha^p}{\partial v_\beta^r} \equiv 0, \quad \frac{\partial z_\alpha^t}{\partial z_\beta^i} \equiv 0,$$

for $p = m + 1, \dots, n$, $r = 1, \dots, m$, $t = m + l + 1, \dots, n$, $i = m + 1, \dots, n$. (Please note that we are *not* following our usual convention.) Looking at the transition functions of the tangent bundle of N_S , in an atlas adapted to S and \mathcal{F} , on the first infinitesimal neighborhood of the embedding of S as the zero section of N_S , we find that the following equality holds:

$$[\delta_j^i]_2 = \left[\frac{\partial v_\alpha^r}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial v_\alpha^r} + \frac{\partial z_\alpha^p}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial z_\alpha^p} \right]_2.$$

Now, since $\partial z_\alpha^p / \partial v_\beta^r \equiv 0$ we have

$$[\delta_j^i]_2 = \left[\frac{\partial z_\alpha^p}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial z_\alpha^p} \right]_2$$

and since $\partial z_\alpha^t / \partial z_\beta^i \equiv 0$ for $t = m + l + 1, \dots, n$, $i = m + 1, \dots, n$ we have

$$[\delta_j^i]_2 = \left[\frac{\partial z_\alpha^{i'}}{\partial z_\beta^j} \frac{\partial z_\beta^i}{\partial z_\alpha^{i'}} \right]_2,$$

where $i, j = m + 1, \dots, m + l$ and we sum over $i' = m + 1, \dots, m + l$.

LEMMA 4.7

Let S be splitting in M , and let \mathcal{F} be a foliation of S ; if the sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{\text{pr}} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0$$

splits on the first infinitesimal neighborhood of S embedded as the zero section of its normal bundle in M , then \mathcal{F} extends as a subsheaf of $\mathcal{T}_{S(1)}$.

Proof

Suppose that we are working in an atlas adapted to S and \mathcal{F} ; on S we have the following isomorphism:

$$(17) \quad \tilde{\mathcal{F}}|_S = \mathcal{V}|_S \oplus \mathcal{F}.$$

This follows directly from our construction of $\tilde{\mathcal{F}}$ as the pullback foliation defined by the integrable system $\{\pi^*(dz_\alpha^{m+l+1}|_S), \dots, \pi^*(dz_\alpha^n|_S)\}$. Therefore we have $\tilde{\mathcal{F}}/\mathcal{V}|_S \simeq \mathcal{F}$, and this implies that the cochain representing (16) vanishes identically when restricted to S . Therefore we know that the components of the cochain $\{U_\alpha, \sigma_\alpha\}$ are identically zero when restricted to S . Let v be a section of $\tilde{\mathcal{F}}/\mathcal{V}$; its image $\tau_\alpha - \sigma_\alpha(v)$ is a section of $\tilde{\mathcal{F}}$. From the discussion above we can remark is that $\sigma_\alpha(v)|_S \equiv 0$; moreover, since on S (17) holds we have $\tau_\alpha(v)|_S \in \mathcal{F} \subset \mathcal{T}_S$, and this proves that v belongs to $\mathcal{T}_{S(1)}$. \square

COROLLARY 4.8

Let S be a splitting submanifold in M , and let \mathcal{F} be a rank 1 foliation of S ; if the sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \tilde{\mathcal{F}} \xrightarrow{\text{pr}} \tilde{\mathcal{F}}/\mathcal{V} \longrightarrow 0$$

splits on the first infinitesimal neighborhood of S embedded as the zero section of its normal bundle in M , then \mathcal{F} extends as a foliation of the first infinitesimal neighborhood of S in M . Moreover, we can find an atlas adapted to S and \mathcal{F} given by a collection of charts $\{U_\alpha, (v_\alpha^1, \dots, v_\alpha^m, z_\alpha^{m+1}, \dots, z_\alpha^n)\}$ such that the class (16) can be represented by the zero cochain.

Proof

If $\tilde{\mathcal{F}}/\mathcal{V}$ has rank 1 we have that its image through the splitting morphism of (14) is a rank 1 (therefore involutive) subbundle of $\mathcal{T}_{S_N(1)}$. Thanks to Lemma 4.4 we have the first part of the assertion. Corollary 1.11 gives us the second part of the assertion. \square

COROLLARY 4.9

Let M be an n -dimensional complex manifold, let S be a codimension m splitting submanifold, and let \mathcal{F} be a regular foliation of S . Suppose that S admits a first-order extendable tangent bundle; then \mathcal{F} extends to a subsheaf of $\mathcal{T}_{S(1)}$.

Proof

The first-order extendable tangent bundle implies the vanishing of (16) and the splitting of sequence (14); the extension of \mathcal{F} is then given by the image of $\tilde{\mathcal{F}}/\mathcal{V}$ in $\tilde{\mathcal{F}}$. \square

REMARK 4.10

The reason why the splitting of (14) is not a sufficient condition for the foliation to extend to the first infinitesimal neighborhood lies in the fact that the image

of $\tilde{\mathcal{F}}/\mathcal{V}$ may not be involutive. If this image is involutive we have a statement similar to the one in the last corollary; anyway, even if it is not involutive, thanks to the results in Section 5, the splitting of (14) is enough to get some important insights on the Khanedani–Lehmann–Suwa action.

REMARK 4.11

We want to see what happens in coordinates when we can extend the foliation. First of all, the vanishing of the class (16) in cohomology means there exists a cochain $\{U_\alpha, \sigma_\alpha\} \in C^0(S_N(1), (\mathcal{F}/\mathcal{V})^* \otimes \mathcal{V})$ such that

$$\sigma_\beta - \sigma_\alpha = - \left[\frac{\partial z_\alpha^j}{\partial z_\beta^i} \frac{\partial v_\beta^r}{\partial z_\alpha^j} \right]_2 \omega_\beta^i \otimes \frac{\partial}{\partial v_\beta^r}.$$

In a coordinate system adapted to S and \mathcal{F} on each U_α we can write the elements of the cochain as

$$\sigma_\alpha = [c_{j,\alpha}^s]_2 \omega_\alpha^j \otimes \frac{\partial}{\partial v_\alpha^s}.$$

Since the sequence splits when it is restricted to S we can assume that the coefficients $c_{j,\alpha}^s$ of each σ_α belong to $\mathcal{I}_S/\mathcal{I}_S^2$. Without loss of generality we can suppose that the local lifts τ_α send the generators of $\tilde{\mathcal{F}}/\mathcal{V}$, which we denote by $\partial_{i,\alpha}$, in the coordinate fields $\partial/\partial z_\alpha^i$. (The difference about two different choices of lifts is absorbed by the cochain.) Then a generator $\partial/\partial z_\alpha^i|_S$ of \mathcal{F} on U_α extends to the section v of $\mathcal{T}_{S(1)}$ given by

$$-[c_{j,\alpha}^s]_2 \frac{\partial}{\partial v_\alpha^s} + \frac{\partial}{\partial z_\alpha^j}.$$

5. Action of subsheaves of \mathcal{F} on $\mathcal{N}_{\mathcal{F},M}$

As usual let \mathcal{F} be a foliation of S : in this section we shall discuss how the existence of coherent subsheaves of $\mathcal{T}_{S(1)}$ that, restricted to S , are subsheaves of \mathcal{F} gives rise to variation actions on $\mathcal{N}_{\mathcal{F},M}$.

LEMMA 5.1

Let \mathcal{E} be a coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S , is a subsheaf of \mathcal{F} . Then \mathcal{E} is a subsheaf of $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$.

Proof

Let $\{U_\alpha, z_\alpha\}$ be an atlas adapted to S and \mathcal{F} . On each coordinate chart, a section v of \mathcal{E} can be written as

$$[a^u]_2 \frac{\partial}{\partial z^u} + [a^i]_2 \frac{\partial}{\partial z^i},$$

with $a^u \in \mathcal{I}_S$. Therefore, thanks to Remark 2.5, we know that v belongs to $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$. \square

DEFINITION 5.2

Let \mathcal{E} be a coherent subsheaf of $\mathcal{T}_{S(1)}$; we say it is S -faithful if the restriction map $|_S : \mathcal{E} \rightarrow \mathcal{E}|_S$ is injective.

PROPOSITION 5.3

Suppose that \mathcal{E} is a coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S , is a subsheaf of \mathcal{F} , that is generated on an open set U_α by $\tilde{v}_{1,\alpha}, \dots, \tilde{v}_{k,\alpha}$, and that is S -faithful. Then there exists a partial holomorphic connection (δ, \mathcal{E}) for $\mathcal{N}_{\mathcal{F},M}$.

Proof

Since there are no generators sent to zero by the restriction to S , then $\mathcal{E}|_{S \cap U_\alpha}$ is generated by $v_{k,\alpha} := \tilde{v}_{k,\alpha}|_S$. Please keep in mind that the generators of $\mathcal{E}|_S$ are always the restriction of the generators of \mathcal{E} , so, choosing the local generators of \mathcal{E} we have a canonical way to extend the local generators of $\mathcal{E}|_S$.

Let π be the projection from $\mathcal{T}_{M,S(1)}^{\mathcal{F}}$ to \mathcal{A} , and let w be a section of $\mathcal{E}|_S$; we define a map $\tilde{\pi} : \mathcal{E}|_S \rightarrow \mathcal{A}$ by $\pi(w) := \pi(\tilde{w})$, where \tilde{w} is an extension of w as a section of \mathcal{E} . On a trivializing neighborhood for \mathcal{E} a section has the following form: $w = [f^k]_1 v_{k,\alpha} \in \mathcal{E}|_{S \cap U_\alpha}$. The difference between two representatives \tilde{w}_1 and \tilde{w}_2 of w in \mathcal{E} on U_α can be written in the following form:

$$[g^k]_2 \tilde{v}_{k,\alpha},$$

where the g^k belong to $\mathcal{I}_S/\mathcal{I}_S^2$ and therefore belong to $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$. Therefore the map $\tilde{\pi}$ does not depend on the extension chosen.

Suppose now that we have a section w of $\mathcal{E}|_S$ and two coordinate charts U_α and U_β on which the section is represented as $w_\alpha = [f_\alpha^k]_1 v_{k,\alpha}$ and $w_\beta = [f_\beta^k]_1 v_{k,\beta}$. Now, we have that, since \mathcal{E} is a subbundle of $\mathcal{T}_{S(1)}$,

$$\tilde{v}_{k,\alpha} = [(h_{\alpha\beta})_k^h]_2 \tilde{v}_{h,\beta},$$

which implies also that

$$[f_\alpha^k (h_{\alpha\beta})_k^h]_1 = [f_\beta^h]_1.$$

We take two extensions \tilde{w}_α and \tilde{w}_β on U_α and U_β , respectively: we claim their difference lies in $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$. We compute

$$\begin{aligned} (\tilde{w}_\beta - \tilde{w}_\alpha)|_S &= ([\tilde{f}_\alpha^k]_2 \tilde{v}_{k,\alpha} - [\tilde{f}_\beta^h]_2 \tilde{v}_{h,\beta})|_S \\ &= ([\tilde{f}_\alpha^k]_2 [h_{\alpha\beta,k}^h]_2 \tilde{v}_{h,\beta} - [\tilde{f}_\beta^h]_2 \tilde{v}_{h,\beta})|_S \\ &= [[f_\alpha^k (h_{\alpha\beta})_k^h]_2 - [f_\beta^h]_2]_1 v_{h,\beta} = [0]_1. \end{aligned}$$

As stated, the difference between the two extensions lies in $\mathcal{I}_S \mathcal{T}_{M,S(1)}^{\mathcal{F}}$. So, the map $\tilde{\pi} : \mathcal{E}|_S \rightarrow \mathcal{A}$ is an \mathcal{O}_S -morphism between $\mathcal{E}|_S$ and \mathcal{A} giving a splitting of the following sequence:

$$0 \longrightarrow \text{Hom}(\mathcal{N}_S, \mathcal{N}_{\mathcal{F},M}) \longrightarrow \mathcal{A}_{\mathcal{F},\mathcal{E}|_S} \xrightarrow{\Theta_1} \mathcal{E}|_S \longrightarrow 0,$$

where $\mathcal{A}_{\mathcal{F},\mathcal{E}|_S}$ is the preimage of $\mathcal{E}|_S$ in \mathcal{A} through Θ_1 . □

Therefore, recalling Section 2 we have that there is a partial holomorphic connection on $\mathcal{N}_{\mathcal{F},M}$ along $\mathcal{E}|_S$, given as follows:

$$\delta_v(s) = \tilde{X}_{\pi(\tilde{v})}(s),$$

where \tilde{X} is the universal connection on $\mathcal{A}_{\mathcal{N}_{\mathcal{F},M}}$.

REMARK 5.4

This connection may not be flat. Therefore we can use the Bott vanishing theorem (see [ABT2, Theorem 6.1]) only in its noninvolutive form.

COROLLARY 5.5

Suppose that \mathcal{E} is an involutive coherent subsheaf of $\mathcal{T}_{S(1)}$ that, restricted to S , is a subsheaf of \mathcal{F} and S -faithful. Then there exists a flat partial holomorphic connection (δ, \mathcal{E}) for $\mathcal{N}_{\mathcal{F},M}$.

Proof

From Proposition 5.3 we already know there exists a partial holomorphic connection along \mathcal{E} ; since \mathcal{E} is involutive we can check if it is flat:

$$\delta_u(\delta_v(s)) - \delta_v(\delta_u(s)) - \delta_{[u,v]}(s) = \text{pr}([\tilde{u}, [\tilde{v}, \tilde{s}]] - [\tilde{v}, [\tilde{u}, \tilde{s}]] - [[\tilde{u}, \tilde{v}], \tilde{s}]) = 0,$$

by the Jacobi identity. □

REMARK 5.6

In the paper [ABT2] is defined the notion of the Lie algebroid morphism; given an involutive coherent subsheaf of $\mathcal{T}_{S(1)}$ the splitting that gives rise to the partial holomorphic connection is a Lie algebroid morphism, and Corollary 5.5 mirrors the fact that the universal partial holomorphic connection is flat (see Proposition 2.13).

COROLLARY 5.7

Suppose that \mathcal{E} is an involutive coherent subsheaf of $\mathcal{T}_{S(1)}$, whose restriction to S is a foliation of S and is S -faithful. Then there exists a flat partial holomorphic connection (δ, \mathcal{E}) for \mathcal{N}_S .

Proof

If we take $\mathcal{F} = \mathcal{T}_S$ in Corollary 5.5 the assertion follows. □

6. Singular holomorphic foliations of the first infinitesimal neighborhood

This section is devoted to making precise what we mean by singular foliations of infinitesimal neighborhoods. In some sense, we want to prove an analogue of the following proposition, stated in [Su] and proved in [MY].

PROPOSITION 6.1

If a foliation is reduced, then $\text{Codim } S(\mathcal{F}) \geq 2$. If \mathcal{F} is locally free and if $\text{Codim } S(\mathcal{F}) \geq 2$, then \mathcal{F} is reduced.

DEFINITION 6.2

A singular foliation of $S(k)$ is a rank l coherent subsheaf (see Definition 0.2) \mathcal{F} of $\mathcal{T}_{S(k)}$, such that

- for every $x \in S$ we have $[\mathcal{F}_x, \mathcal{F}_x] \subseteq \mathcal{F}_x$ (where the bracket is the one defined in Lemma 1.5);
- the restriction $\mathcal{F}|_S$ (see Definition 1.2) is a rank l singular foliation of S (see [Su, Definition 1.1]).

DEFINITION 6.3

Let \mathcal{F} be a singular holomorphic foliation of $S(k)$. We set $\mathcal{N}_{\mathcal{F}} = \mathcal{T}_{S(k)}/\mathcal{F}$, and we denote by $S(\mathcal{F}) := \text{Sing}(\mathcal{N}_{\mathcal{F}})$ the *singular set of the foliation*.

DEFINITION 6.4

Let \mathcal{F} be a singular foliation of $S(k)$. We say that \mathcal{F} is *reduced* if it is full in $\mathcal{T}_{S(k)}$; that is, for any open set U in S we have

$$\Gamma(U, \mathcal{T}_{S(k)}) \cap \Gamma(U \setminus S(\mathcal{F}), \mathcal{F}) = \Gamma(U, \mathcal{F}).$$

LEMMA 6.5

Let \mathcal{F} be a singular foliation of $S(k)$; then there exists a canonical way to associate to it a reduced singular foliation of $S(k)$.

Proof

We cover now a neighborhood of S by open sets $\{U_\alpha\}$ such that $\mathcal{F}_{U_\alpha \cap S}$ is generated by $v_{1,\alpha}, \dots, v_{l,\alpha}$ and on each U_α we can extend the $v_{i,\alpha}$ to logarithmic vector fields $\tilde{v}_{i,\alpha}$ on U_α . On U_α the $\tilde{v}_{i,\alpha}$ define a distribution with a sheaf of sections \mathcal{D}_α ; please note that this is a sheaf on U_α , not on the whole M . We define $\mathcal{N}_{\mathcal{D}_\alpha} = \mathcal{T}_M|_{U_\alpha}/\mathcal{D}_\alpha$ and denote by $S(\mathcal{D}_\alpha)$ the singularity set of $\mathcal{N}_{\mathcal{D}_\alpha}$. In general, this distribution may not be reduced, that is, $\Gamma(U_\alpha, \mathcal{T}_M) \cap \Gamma(U_\alpha \setminus S(\mathcal{D}_\alpha), \mathcal{D}_\alpha) \neq \Gamma(U_\alpha, \mathcal{D}_\alpha)$. We take now the annihilator $(\mathcal{D}_\alpha)^a = \{\omega \in \Omega_M \mid \omega(v) = 0 \text{ for every } v \in \mathcal{D}_\alpha\}$. If we take its annihilator $\tilde{\mathcal{D}}_\alpha := ((\mathcal{D}_\alpha)^a)^a = \{w \in \mathcal{T}_M \mid \omega(w) = 0 \text{ for every } \omega \in (\mathcal{D}_\alpha)^a\}$ we get now a reduced sheaf of sections of the distribution, generated by sections $\tilde{w}_{1,\alpha}, \dots, \tilde{w}_{l,\alpha}$; we can take the same l because, since we are dealing with coherent sheaves, the rank is constant outside the singularity set.

Since $\Gamma(U_\alpha, \mathcal{D}_\alpha) \subset \Gamma(U_\alpha, \tilde{\mathcal{D}}_\alpha)$ we have $\tilde{v}_{i,\alpha} = (h_\alpha)_i^j \tilde{w}_{j,\alpha}$, where $(h_\alpha)_i^j$ is a matrix of holomorphic functions that may be singular on a subset of U_α of codimension smaller than 2, contained in $S(\mathcal{D}_\alpha)$. We remark also that $S(\mathcal{F}) \subset S(\mathcal{D}_\alpha)$ and that the $\tilde{w}_{i,\alpha}$ are logarithmic vector fields.

We want to check now that $\tilde{\mathcal{D}}_\alpha \otimes \mathcal{O}_{S(k)}|_{(U_\alpha \cap S) \setminus S(\mathcal{F})}$ generates \mathcal{F} and is involutive. We will denote the restriction of $\tilde{w}_{i,\alpha}$ to the k th infinitesimal neighborhood

by $w_{i,\alpha}$. Indeed, outside the singularity set, the matrix $(h_\alpha)_i^j$ is invertible as a matrix of holomorphic functions, with inverse $(g_\alpha)_j^i$ which implies that the $w_{i,\alpha}$'s generate \mathcal{F} . We check the involutivity:

$$\begin{aligned} [\tilde{w}_{i,\alpha}, \tilde{w}_{i',\alpha}] \otimes [1]_{k+1} &= [(g_\alpha)_i^j \tilde{v}_{j,\alpha}, (g_\alpha)_{i'}^{j'} \tilde{v}_{j',\alpha}] \otimes [1]_{k+1} \\ &= [(g_\alpha)_i^j]_{k+1} v_{j,\alpha} ([(g_\alpha)_{i'}^{j'}]_{k+1}) v_{j',\alpha} \\ &\quad - [(g_\alpha)_{i'}^{j'}]_{k+1} v_{j',\alpha} ([(g_\alpha)_i^j]_{k+1}) v_{j,\alpha} \\ &\quad + [(g_\alpha)_{i'}^{j'}]_{k+1} [(g_\alpha)_i^j]_{k+1} [v_{j,\alpha}, v_{j',\alpha}]. \end{aligned}$$

Note that $(g_\alpha)_i^j$ is a matrix of meromorphic functions on U_α (this follows from the Cramer rule for the inverse of a matrix), and its inverse is a matrix of holomorphic functions. Now, for each $v_{j,\alpha}$ we have

$$v_{j,\alpha} ([(g_\alpha)_{i'}^{j'}]_{k+1}) = - [(g_\alpha)_{i'}^{j''}]_{k+1} v_{j,\alpha} ([(h_\alpha)_{j''}^{i''}]_{k+1}) [(g_\alpha)_{i''}^{j'}]_{k+1},$$

and therefore

$$\begin{aligned} & [(g_\alpha)_i^j]_{k+1} v_{j,\alpha} ([(g_\alpha)_{i'}^{j'}]_{k+1}) v_{j',\alpha} \\ &= - [(g_\alpha)_i^j]_{k+1} [(g_\alpha)_{i'}^{j''}]_{k+1} v_{j,\alpha} ([(h_\alpha)_{j''}^{i''}]_{k+1}) [(g_\alpha)_{i''}^{j'}]_{k+1} v_{j',\alpha} \\ &= - [(g_\alpha)_{i'}^{j''}]_{k+1} w_{i,\alpha} ([(h_\alpha)_{j''}^{i''}]_{k+1}) w_{i'',\alpha}. \end{aligned}$$

A similar reasoning holds for the second summand in the involutivity check. If we denote by $[a_{j,j'}^{j''}]_{k+1}$ the elements of $\mathcal{O}_{S(k)}$ such that

$$[v_{j,\alpha}, v_{j',\alpha}] = [a_{j,j'}^{j''}]_{k+1} v_{j'',\alpha}$$

we have

$$\begin{aligned} & [(g_\alpha)_{i'}^{j'}]_{k+1} [(g_\alpha)_i^j]_{k+1} [v_{j,\alpha}, v_{j',\alpha}] \\ &= - [(g_\alpha)_{i'}^{j'}]_{k+1} [(g_\alpha)_i^j]_{k+1} [a_{j,j'}^{j''}]_{k+1} v_{j'',\alpha} \\ &= - [(g_\alpha)_{i'}^{j'}]_{k+1} [(g_\alpha)_i^j]_{k+1} [a_{j,j'}^{j''}]_{k+1} [(h_\alpha)_{j''}^{i''}]_{k+1} w_{i'',\alpha}. \end{aligned}$$

The point these computations prove is that $[\tilde{w}_{i,\alpha}, \tilde{w}_{i',\alpha}] \otimes [1]_{k+1}$ belongs to the module generated by the $w_{i,\alpha}$'s over the meromorphic functions. But, a priori, we know that this bracket is a holomorphic section of $\mathcal{T}_{S(k)}$ and therefore belongs to the $\mathcal{O}_{S(k)}$ -module generated by the $w_{i,\alpha}$'s. □

REMARK 6.6

By the proof of Lemma 6.5 and by Proposition 6.1 we have that each one of the extensions $\tilde{w}_{i,\alpha}$ has a singularity set of codimension at least 2.

7. Index theorems for foliations and involutive closures

Following the work [Su] and the articles [ABT1] and [ABT2], we know that the existence of a partial holomorphic connection gives rise to the vanishing of some of the Chern classes of a vector bundle and therefore to an index theorem

thanks to Bott’s vanishing theorem (see [Su, Theorem 9.11, p. 76] for the version for virtual bundles and [ABT2, Theorem 6.1] for the version for noninvolutive subbundles).

In Section 2 we found a concrete realization of the Atiyah sheaf for the normal bundle of a foliation as a quotient of the ambient tangent bundle, and we proved that the Atiyah sequence splits if there exists a foliation of the first infinitesimal neighborhood. In this section we state the index theorems that follow directly from our treatment.

The proofs for all the points of the theorem follow from the fact that the existence of a partial holomorphic connection implies the vanishing of the characteristic classes, following the general theory of [Su], [ABT1], [ABT2], and [ABT3]. Therefore, from the theory developed in the previous sections, we have the following theorem.

THEOREM 7.1

Suppose alternatively that

- (1) *there exists a rank l foliation \mathcal{F} on S , such that it extends to the first infinitesimal neighborhood of $S \setminus S(\mathcal{F})$;*
- (2) *there exists a foliation \mathcal{F} on S and a rank l subsheaf \mathcal{E} of $\mathcal{T}_{S(1)}$ that, restricted to S , is a subsheaf of \mathcal{F} , and is S -faithful;*
- (3) *there exists a rank l holomorphic foliation \mathcal{F} defined on a neighborhood of S and a 2-splitting, first-order \mathcal{F} -faithful outside an analytic subset Σ of U containing $S(\mathcal{F}) \cap S$, $S \not\subseteq \Sigma$;*
- (4) *there exists a rank l holomorphic foliation \mathcal{F} defined on S and that sequence (14) splits;*
- (5) *there exists a rank l holomorphic foliation \mathcal{F} defined on S and that sequence (14) splits and the image $\tilde{\mathcal{F}}/\mathcal{V}$ in $\tilde{\mathcal{F}}$ is involutive;*
- (6) *S is splitting and admits a first-order extendable tangent bundle and there exists \mathcal{F} , a rank l holomorphic foliation defined of S .*

Let $\Sigma = S(\mathcal{F})$ (resp., $\Sigma = S(\mathcal{F}) \cap S(\mathcal{E})$ in (2)), let $\mathcal{G} = \mathcal{F}$ (resp., $\mathcal{G} = \mathcal{E}$ in (2)), and let $\Sigma = \bigcup_{\lambda} \Sigma_{\lambda}$ be the decomposition of Σ in connected components. By abuse of notation denote by $\mathcal{N}_{\mathcal{F}^{\sigma}, M}$ the sheaf $\mathcal{N}_{\mathcal{F}, M}$ even if \mathcal{F} is tangent to S . Then for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l$ (resp., $n - m - l + \lfloor l/2 \rfloor$ in (2), (5), (6) if the sheaf along which we construct the partial holomorphic connection is not involutive) we can define the residue $\text{Res}_{\phi}(\mathcal{G}, \mathcal{N}_{\mathcal{F}^{\sigma}, M}; \Sigma_{\lambda}) \in H_{2(n-m-k)}(\Sigma_{\lambda})$ depending only on the local behaviour of \mathcal{G} and $\mathcal{N}_{\mathcal{F}^{\sigma}, M}$ near Σ_{λ} such that

$$\sum_{\lambda} \text{Res}_{\phi}(\mathcal{G}, \mathcal{N}_{\mathcal{F}^{\sigma}, M}; \Sigma_{\lambda}) = \int_S \phi(\mathcal{N}_{\mathcal{F}^{\sigma}, M}),$$

where $\phi(\mathcal{N}_{\mathcal{F}^{\sigma}, M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{F}^{\sigma}, M}$.

Another interesting result following from our theory is obtained by defining, for a coherent subsheaf \mathcal{E} of $\mathcal{T}_{S(1)}$, a natural object, its involutive closure, the

smallest involutive subsheaf containing \mathcal{E} . Thanks to the machinery developed in Section 5, it is proved that the existence of \mathcal{E} gives rise to vanishing theorems for its involutive closure.

DEFINITION 7.2

Let \mathcal{E} be a coherent subsheaf of $\mathcal{T}_{S(1)}$ such that $\mathcal{E}|_S$ is nonempty. We denote by $\text{Sing}(\mathcal{E})$ the set $\{x \in S \mid \mathcal{T}_{S(1)}/\mathcal{E} \text{ is not } \mathcal{O}_{S(1),x}\text{-free}\}$. On $S \setminus \text{Sing}(\mathcal{E})$ we define the *involutive closure* \mathcal{G} of \mathcal{E} in S to be the intersection of all the coherent involutive subsheaves of \mathcal{T}_S containing $\mathcal{E}|_S$.

Recall that the intersection of coherent subsheaves of \mathcal{T}_S is again a coherent subsheaf of \mathcal{T}_S ; now, \mathcal{G} is involutive by definition and therefore gives rise to a foliation of S . Clearly, $\mathcal{E}|_S$ is a subsheaf of \mathcal{G} , and we can apply Proposition 5.3, getting the following result.

THEOREM 7.3

Suppose that \mathcal{E} is a coherent subsheaf of $\mathcal{T}_{S(1)}$ of rank l , whose restriction $\mathcal{E}|_S$ has rank l . Let \mathcal{G} be the involutive closure of \mathcal{E} in S . Let $\Sigma = S(\mathcal{E}) \cup S(\mathcal{G}) = \bigcup_{\alpha} \Sigma_{\alpha}$ be the decomposition of Σ in connected components. Then, for every symmetric homogeneous polynomial ϕ of degree k larger than $n - m - l + \lfloor l/2 \rfloor$ we can define the residue $\text{Res}_{\phi}(\mathcal{E}|_S, \mathcal{N}_{\mathcal{G},M}; \Sigma_{\alpha}) \in H_{2(n-m-k)}(\Sigma_{\alpha})$ depending only on the local behaviour of $\mathcal{E}|_S$ and $\mathcal{N}_{\mathcal{G},M}$ near Σ_{α} such that:

$$\sum_{\lambda} \text{Res}_{\phi}(\mathcal{E}|_S, \mathcal{N}_{\mathcal{G},M}; \Sigma_{\alpha}) = \int_S \phi(\mathcal{N}_{\mathcal{G},M}),$$

where $\phi(\mathcal{N}_{\mathcal{G},M})$ is the evaluation of ϕ on the Chern classes of $\mathcal{N}_{\mathcal{G},M}$.

8. Computing the residue in the simplest case

In this section we compute the residue for a codimension 1 foliation of the first infinitesimal neighborhood of a codimension 1 submanifold in a surface. Let (U_1, x, y) be a neighborhood of zero in \mathbb{C}^2 , let $S = \{x = 0\}$, let \mathcal{F} be a foliation of $S(1)$ such that $\text{Sing}(\mathcal{F}) = \{0\}$, and let v be a generator of \mathcal{F} , that is, a holomorphic section of $\mathcal{T}_{S(1)}$ with an isolated singularity in zero. Supposing \mathcal{F} reduced, from Section 6 and Remark 6.6 we see that this assumption does not give rise to a loss of generality for our computation.

REMARK 8.1

Please note also that, if we denote by \tilde{v} an extension of v to U_1 and by $\tilde{\mathcal{F}}$ the foliation generated by it, thanks to how we defined the holomorphic action and the theory developed for local extensions, the computation of this residue could be reduced to the computation of the residue given by the Lehmann–Khanedani–Suwa action of \tilde{v} on $\mathcal{N}_{\mathcal{F}}|_S$, which can be found, for example, in [Su, Chapter IV, Theorem 5.3].

We will, anyway, compute the index explicitly in the framework we developed. Call $U_0 := U_1 \setminus \{0\}$; with an abuse of notation we will also say $M := U_1$. Let G be the trivial line bundle on S ; we can see $v|_S$ as a holomorphic homomorphism between G and TS . On U_0 we can see G as a subbundle of $TM|_S$; moreover G embedded through $v|_S$ is nothing else but the bundle associated to $\mathcal{F}|_S$. Therefore, we can speak of the virtual bundle $[TM|_S - G]$, which coincides, on U_0 , with the normal bundle to the foliation $\mathcal{F}|_{U_0 \cap S}$ in the ambient tangent bundle $TM|_{U_0 \cap S}$, denoted by $N_{\mathcal{F},M}$. Since the only homogeneous symmetric polynomial in dimension 1 is the trace, we would like to compute the residue for the first Chern class of $[TM|_S - G]$, whose sheaf of sections is $\mathcal{N}_{\mathcal{F},M}$. The first Chern class being additive, we are going to compute $c_1(TM|_S) - c_1(G)$. If U_0 is small enough, thanks to the embedding of G into $TM|_S$ we have that on U_0 we can see $TM|_S$ as the direct sum $G \oplus N_{\mathcal{F},M}$. We are going to apply [Su, Proposition 8.4, p. 73] to the following sequence:

$$0 \longrightarrow F|_{U_0 \cap S} \longrightarrow TM|_{U_0 \cap S} \longrightarrow N_{\mathcal{F},M} \longrightarrow 0.$$

We want to build on U_0 a family of connections compatible with the sequence, so that the Bott vanishing theorem for virtual bundles (see [Su, Theorem 9.11, p. 76]) implies that $c_1(N_{\mathcal{F},M})$ on U_0 is zero. We proved that the existence of a foliation of the first infinitesimal neighborhood gives rise to partial connection on $N_{\mathcal{F},M}$. Now, thanks to Corollary 2.15 we can compute the actual connection matrix of this partial holomorphic connection on $\mathcal{N}_{\mathcal{F},M}$ and extend it to a connection on $N_{\mathcal{F},M}$, denoted by ∇ . To build a family of connections simplifying our computations we take on $U_0 \cap S$ the connection ∇_0^G which is trivial with respect to the generator 1_G of the trivial line bundle G . Since $TM|_S$ on $U_0 \cap S$ is the direct sum of G and $N_{\mathcal{F},M}$ we let the connection for $TM|_S$ be the direct sum connection $\nabla_0^{TM} := \nabla \oplus \nabla_0^G$. Both ∇_0^{TM} and ∇_0^G are holomorphic connections along F ; therefore we can apply Bott's vanishing theorem in the version for virtual bundles and obtain $c_1(N_{\mathcal{F},M}) \equiv 0$ on U_0 .

In Čech–de Rham cohomology relative to the cover $\{U_0, U_1\}$ the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ is represented as a triple $(\omega_0, \omega_1, \sigma_{01})$, where ω_0 is the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ on U_0 and ω_1 is the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ on U_1 , while σ_{01} is a 1-form, the Bott difference form, that is, a 1-form such that $\omega_1 - \omega_0 = d\sigma_{01}$ on $U_0 \cap U_1$ (for a complete treatment, refer to [Su]). Due to the additivity of the first Chern class, to compute the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ we need to compute the first Chern classes of G and $TM|_S$ on U_1 (we already know that the first Chern class of $\mathcal{N}_{\mathcal{F},M}$ on U_0 is zero) and the Bott difference forms $c_1(\nabla_0^{TM}, \nabla_1^{TM})$ and $c_1(\nabla_0^G, \nabla_1^G)$. On U_1 we can take, again, as a connection for G the connection which is trivial with respect to the generator 1_G of G : therefore $c_1(\nabla_0^G, \nabla_1^G) = 0$, since the connections for G on U_0 and U_1 are the same. On U_1 we take as ∇_1^{TM} the $\partial/\partial x, \partial/\partial y$ trivial connection; then $c_1(\nabla_1^{TM}) = 0$ and the problem reduces to computing the Bott difference form $c_1(\nabla_0^{TM}, \nabla_1^{TM})$. To compute it we need the connection matrix for ∇_0^{TM} with respect to the frame $\partial/\partial x, \partial/\partial y$. First of all we compute the action of ∇ on the equivalence class $\nu = [\partial/\partial x]$ in $N_{\mathcal{F},M}$. The

generator v of \mathcal{F} is written in coordinates as

$$[A]_2 \frac{\partial}{\partial x} + [B]_2 \frac{\partial}{\partial y},$$

where $[A]_2$ belongs to $\mathcal{I}_S/\mathcal{I}_S^2$. In the following, we shall denote by v_S the restriction of \tilde{v} to S ; in coordinates we have $v_S = [B]_1 \partial/\partial y$. We compute now the action of F on $\mathcal{N}_{\mathcal{F},M}$, recalling Corollary 2.15:

$$\nabla_{v_S}(\nu) = \text{pr}\left(\left[[A]_2 \frac{\partial}{\partial x} + [B]_2 \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]\Big|_S\right) = -\left[\frac{\partial A}{\partial x}\right]_1 \nu.$$

We compute now the connection matrix for ∇ . Since

$$-\left[\frac{\partial A}{\partial x}\right]_1 = ([C]_1 \cdot dx + [D]_1 \cdot dy)\left([B]_1 \frac{\partial}{\partial y}\right) = [D \cdot B]_1,$$

it follows that the connection matrix is nothing else but

$$\omega = -\left[\frac{\partial A}{\partial x} \frac{1}{B}\right]_1 dy.$$

We have now all the tools needed to compute the connection matrix for ∇_0^{TM} :

$$\begin{aligned} \nabla_0^{TM}\left(\frac{\partial}{\partial x}\right) &= \nabla(\nu) = -\left[\frac{\partial A}{\partial x} \cdot \frac{1}{B}\right]_1 dy \otimes \frac{\partial}{\partial x}, \\ \nabla_0^{TM}\left(\frac{\partial}{\partial y}\right) &= \nabla_0^G\left(\frac{1}{B} \cdot v\right) = -\left[\frac{dB}{B^2}\right]_1 \cdot v = -\left[\frac{dB}{B}\right]_1 \otimes \frac{\partial}{\partial y}. \end{aligned}$$

Thus the connection matrix has the following form:

$$\begin{bmatrix} -\left[\frac{\partial A}{\partial x} \frac{1}{B}\right]_1 dy & 0 \\ 0 & -\left[\frac{dB}{B}\right]_1 \end{bmatrix}.$$

Considering the bundle $TM \times [0, 1] \rightarrow M \times [0, 1]$ and the connection $\tilde{\nabla}$ given by $\tilde{\nabla} := (1 - t)\nabla_0^{TM} + t\nabla_1^{TM}$ we can compute the Bott difference form given by $\pi_*(c_1(\tilde{\nabla}))$ where π_* is integration along the fiber of the projection $\pi : M \times [0, 1] \rightarrow M$. The Bott difference form is then

$$\left[\frac{1}{B} \frac{\partial A}{\partial x}\right]_1 dy + \left[\frac{dB}{B}\right]_1.$$

So, the residue for $c_1(\mathcal{N}_{\mathcal{F},M})$ in zero is

$$\frac{1}{2\pi\sqrt{-1}} \int_{\{x=0, |y|=\varepsilon\}} \left[\frac{1}{B} \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right)\right]_1 dy.$$

9. The residue for the simplest transversal case

Let (U_1, x, y) be a neighborhood of zero in \mathbb{C}^2 , and let $S = \{x = 0\}$. Let now v be a holomorphic section of $\mathcal{T}_{M,S(1)}$ with an isolated singularity in zero. As before, we say $U_0 := U_1 \setminus \{0\}$ and $M = U_1$. Please note that we drop the hypothesis about v belonging to $\mathcal{T}_{S(1)}$. We want to compute the variation index for such a foliation. Since the situation is local we can assume that we have a local 2-splitting, first-order \mathcal{F} -faithful outside zero and that we are in a chart adapted to it and therefore we have a map $\mathcal{T}_{M,S(1)}$ to $\mathcal{T}_{S(1)}$. Write \tilde{v} in coordinates as

$$\tilde{v} = [A(x, y)]_2 \frac{\partial}{\partial x} + [B(x, y)]_2 \frac{\partial}{\partial y}.$$

Now we can write $[A(x, y)]_2 = [\tilde{\rho}([A(x, y)]_2) + R(x, y)]_2$, where $\tilde{\rho}$ is the θ_1 -derivation associated to the 1-splitting induced by the 2-splitting; then,

$$\sigma^*(\tilde{v}) = (\tilde{\rho}([A(x, y)]_2))\partial/\partial x + B(x, y)\partial/\partial y.$$

Moreover, we have a splitting $\sigma^* : \mathcal{T}_{M,S} \rightarrow \mathcal{T}_S$, giving rise on $U_0 \cap S$ to an isomorphism between \mathcal{F}_S , the sheaf of germs of sections of the foliation generated by $v_S := v|_S$ and the sheaf of germs of sections of \mathcal{F}^σ . Now, the vector field

$$w = [\tilde{\rho}([A(x, y)]_2)]_2 \partial/\partial x + [B(x, y)]_2 \partial/\partial y$$

is a section of $\mathcal{T}_{S(1)}$, giving rise to a foliation of the first infinitesimal neighborhood. We can now compute the index as in Section 8: the residue for $c_1(N_{\mathcal{F}^\sigma, M})$ is therefore

$$\begin{aligned} & \frac{1}{2\pi\sqrt{-1}} \int_{\{|y|=\varepsilon\}} \left[\frac{1}{B} \left(\frac{\partial[\tilde{\rho}([A]_2)]_2}{\partial x} + \frac{\partial B}{\partial y} \right) \right]_1 dy \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\{|y|=\varepsilon\}} \left[\frac{1}{B} \left(\frac{\partial}{\partial x} \left(\frac{\partial A}{\partial x} \cdot x \right) + \frac{\partial B}{\partial y} \right) \right]_1 dy \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\{|y|=\varepsilon\}} \left[\frac{1}{B} \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \right) \right]_1 dy. \end{aligned}$$

REMARK 9.1

The term

$$\frac{\partial^2 A}{\partial x^2} \cdot x$$

in the last computation disappears since it belongs to \mathcal{I}_S .

10. A couple of remarks about extendability of foliations

In this short section we summarize some of the results of this paper, stressing their importance towards the understanding of the following problem: “When is it possible to extend a holomorphic foliation on a submanifold S of codimension m in a complex manifold M to a neighborhood of S ?” Thanks to Theorem 2.10 we know that, if there exists a rank l foliation of the first infinitesimal neighborhood and if we take any symmetric polynomial ϕ of degree larger than $n - m - l$, then $\phi(\mathcal{N}_{\mathcal{F}, M})$ vanishes. Given a foliation \mathcal{F} on S , the classes $\phi(\mathcal{N}_{\mathcal{F}, M})$ are obstructions to finding an extension to the first infinitesimal neighborhood, where ϕ is a symmetric polynomial of degree larger than $n - m - l$. In the splitting case we have much more information. As a matter of fact, if the sequence

$$0 \rightarrow \mathcal{V} \rightarrow \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}/\mathcal{V} \rightarrow 0$$

splits on the first infinitesimal neighborhood of the zero section of \mathcal{N}_S we know that \mathcal{F} can be extended in a noninvolutive way. Therefore, if S splits, the characteristic classes $\phi(\mathcal{N}_{\mathcal{F}, M})$ with ϕ a symmetric polynomial of degree larger than

$n - m - l + \lfloor l/2 \rfloor$ are obstructions to finding an extension of \mathcal{F} as a noninvolutive subbundle of $\mathcal{T}_{S(1)}$. If the extension is involutive, also the characteristic classes $\phi(\mathcal{N}_{\mathcal{F},M})$ with ϕ a symmetric polynomial of degree larger $n - m - l$ and smaller than $n - m - l + \lfloor l/2 \rfloor$ vanish. Therefore, in the splitting case, where it is known that there is a noninvolutive extension, the classes $\phi(\mathcal{N}_{\mathcal{F},M})$ where ϕ is a symmetric polynomial of degree larger $n - m - l$ and smaller than $n - m - l + \lfloor l/2 \rfloor$ are obstructions to finding an involutive extension.

Another interesting remark can follow from a simple example; we look at the procedure we built in Section 4 to extend a foliation in the case where we have a rank 1 foliation \mathcal{F} of a codimension 1 splitting submanifold S in a complex surface M . Thanks to Remark 4.11 we have that the local generators of the extension to the first infinitesimal neighborhood of the foliation \mathcal{F} on M are given on each U_α (modulo rescaling) by

$$\frac{\partial}{\partial z_\alpha^2} - [c_\alpha]_2 \frac{\partial}{\partial z_\alpha^1}.$$

As expected, recalling the computation of the residue in Section 8 we see that, if \mathcal{F} has an isolated singular point in U_α , the computation of the residue depends on the function c_α .

Extending a holomorphic foliation is an important global problem; we have shown that this problem is strictly connected with the residues and the characteristic classes of $\mathcal{N}_{\mathcal{F},M}$.

Acknowledgments. The article is part of the author's Ph.D. dissertation work and he would like to thank Professor Marco Abate, his advisor, for his thoughtful guidance and useful advice and hints; Professor Tatsuo Suwa for many precious conversations, his patience, and his wisdom; and Professor Filippo Bracci for an important suggestion. The author would like to thank Professor Kyoji Saito, the Institute for the Physics and Mathematics of the Universe, Kashiwa, Japan, and the International Centre for Theoretical Physics, Trieste, Italy for the warm hospitality and wonderful research environment offered to him. The author would like also to thank Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni, for the help in funding his mission to Japan.

References

- [ABT1] M. Abate, F. Bracci, and F. Tovena, *Index theorems for holomorphic self-maps*, Ann. of Math. (2) **159** (2004), 819–864.
- [ABT2] ———, *Index theorems for holomorphic maps and foliations*, Indiana Univ. Math J. **57** (2008), 2999–3048.
- [ABT3] ———, *Embeddings of submanifolds and normal bundles*, Adv. Math. **220** (2009), 620–656.
- [Ati] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 182–207.

- [Br] F. Bracci, *First order extensions of holomorphic foliations*, Hokkaido Math. J. **33** (2004), 473–490.
- [Bru] M. Brunella, *Some remarks on indices of holomorphic vector fields*, Publ. Mat. **41** (1997), 527–544.
- [Ca] C. Camacho, “Dicritical singularities of holomorphic vector fields” in *Laminations and Foliations in Dynamics, Geometry and Topology (Stony Brook, N. Y., 1998)*, Contemp. Math. **269**, Amer. Math. Soc., Providence, 2001, 39–45.
- [CL] C. Camacho and D. Lehmann, *Residues of holomorphic foliations relative to a general submanifold*, Bull. London Math. Soc. **37** (2005), 435–445.
- [CMS] C. Camacho, H. Movasati, and P. Sad, *Fibered neighborhoods of curves in surfaces*, J. Geom. Anal. **13** (2003), 55–66.
- [CS] C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math. (2) **115** (1982), 579–595.
- [Ei] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Grad. Texts in Math. **150**, Springer, New York, 1995.
- [Gro] A. Grothendieck, *A general theory of Fibre Spaces With Structure Sheaf*, preprint, 2nd. ed., 1958, www.math.jussieu.fr/~leila/grothendieckcircle/GrothKansas.pdf
- [Ho] T. Honda, *Tangential index of foliations with curves on surfaces*, Hokkaido Math. J. **33** (2004), 255–273.
- [KS] B. Khanedani and T. Suwa, *First variations of holomorphic forms and some applications*, Hokkaido Math. J. **26** (1997), 323–335.
- [Lee] J. M. Lee, *Introduction to Smooth Manifolds*, Grad. Texts in Math. **218**, Springer, New York, 2003.
- [LS1] D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*, J. Differential Geom. **42** (1995), 165–192.
- [LS2] ———, *Generalization of variations and Baum–Bott residues for holomorphic foliations on singular varieties*, Internat. J. Math. **10** (1999), 367–384.
- [MS] J. W. Milnor and J. Stasheff, *Characteristic Classes*, Princeton Univ. Press, Princeton, 1957.
- [MY] Y. Mitera and J. Yoshizaki, *The local analytical triviality of a complex analytic singular foliation*, Hokkaido Math. J. **33** (2004), 275–297.
- [Sa] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 265–291.
- [Su] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Math., Hermann, Paris, 1998.