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To cite this article: A Blumenthal and I Nisoli 2022 *Nonlinearity* **35** 5481

IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **35** (2022) 5481–5504 https://doi.org/10.1088/1361-6544/ac87e8

Noise induced order for skew-products over a non-uniformly expanding base

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Received 24 September 2021, revised 29 July 2022
Accepted for publication 8 August 2022
Published 13 September 2022

Abstract

Noise-induced order (NIO) is the phenomenon by which the chaotic regime of a deterministic system is destroyed in the presence of noise. In this manuscript, we establish NIO for a natural class of systems of dimension $\geqslant 2$ consisting of a fiber-contracting skew product a over nonuniformly-expanding one-dimensional system.

Keywords: random dynamical systems, noise induced order, contracting Lorenz flow, non-uniformly hyperbolic, Lyapunov exponents

Mathematics Subject Classification numbers: 37H15, 37D25, 37D45, 37C30.

(Some figures may appear in colour only in the online journal)

1. Introduction

In recent times, interest in random dynamical systems (RDS) has been greatly stimulated due to their use in applications and their scientific relevance in modeling systems driven by external or internal sources of noise. It is of both theoretical and practical interest to understand the ways in which dynamical behavior can change under the influence of noise. Notable examples include stochastic resonance phenomena (e.g., [8, 15] or the recent mathematical work [11]), where a

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Recommended by Dr Mark F Demers.

stable system can be excited in the presence of noise into producing oscillatory behavior; and noise-induced chaos, where a sufficient amount of noise can induce chaotic behavior in the random dynamics (e.g., [9, 16, 17]).

The topic of this manuscript is noise-induced order (NIO), referring to scenarios where the presence of noise induces stabilization in a previously chaotic deterministic system, quantitatively measured through a transition of the top Lyapunov exponent from positive to negative as the noise amplitude increases. This surprising phenomenon was first observed by numerical experiments in a one dimensional model of the Belosouv–Zhabotinsky reaction [21]. A mathematical proof of this phenomenon was given only recently in [13] via computer-assistance. The recent paper [29] describes a sufficient condition for the existence of NIO in one dimensional non-uniformly expanding systems.

Our purpose here is to establish sufficient conditions for the existence of noise induced order in fiber-contracting skew-products over a non-uniformly hyperbolic base dynamics in the presence of additive noise. As we show here, our abstract framework applies to a fundamental model in dynamics: the Poincaré map of a transverse section for the contracting geometric Lorenz flow, which we refer to hereafter as the *contracting Lorenz two-dimensional map*.

The classical Lorenz model [20, 33] is an important example in nonlinear sciences and a prototypical example of deterministic chaotic behavior. A related model, the so-called geometric Lorenz flow [18], was constructed so as to capture qualitative features of the Poincaré map for the Lorenz model at a natural transversal section. Like the classical Lorenz flow, geometric Lorenz flows admit a saddle equilibrium at the origin, but due to the flexibility of their construction it is possible to parametrically adjust features of these models such as the eigenvalues at the origin. Contracting Lorenz flow refers to these geometric models when the contracting eigenvalues ‘dominate’ the expanding eigenvalue.

Contracting Lorenz flow has been extensively studied. Metzger and Morales proved its stochastic stability [25, 27], while Alves and Soufi proved statistical stability of the associated Poincaré maps [2]. Galatolo, Nisoli and Pacifico proved that the two-dimensional map at a Rovella parameter exhibits exponential decay of correlations with respect to Lipschitz observables [14]. A thermodynamic formalism for contracting Lorenz maps was developed by Pacifico and Todd [30]. Recent works of Alves–Khan and Araujo proved, respectively, that the contracting Rovella flow, a perturbed variant of contracting Lorenz flow, is not statistically stable if we consider all the perturbations in the C^3 topology [1], but is statistically stable if we consider perturbations inside the family of so-called Rovella parameters [4].

Our result on NIO for the contracting Lorenz map leads us to conjecture that the contracting Lorenz flow itself exhibits NIO. The proof of this result, if true, would likely require computer-assisted tools.

Plan for the paper. In section 2 we formulate an abstract framework of fiber-contracting skew products and formally state our result on NIO. The proof of our main result is given in section 3, as well as our application to the contracting Lorenz map. Finally, in section 4 we provide an outlook and some concluding remarks, including a conjecture regarding NIO for the contracting Lorenz flow. Section 5 contains a brief appendix on Lyapunov exponents and the multiplicative ergodic theorem.

2. Setting and statement of results

Basic setup. Throughout, we consider random perturbations of a fixed deterministic skew product map $F : [-1, 1]^{d+1} \circlearrowleft$ exhibiting nonuniform hyperbolicity, where $d \geqslant 1$. The mapping F

is of the form

$$F(x, y) = (T(x), G(x, y)),$$

where $T : [-1, 1] \circlearrowleft$ (the *base dynamics*) and $G : [-1, 1] \times [-1, 1]^d \to [-1, 1]^d$ (the *fiber dynamics*) are mappings on their respective domains. Precisely:

(a) $x \mapsto T(x)$ is piecewise[5] smooth with respect to a finite partition of $[-1, 1]$ into disjoint intervals, while $\{T' = 0\}$ is finite and $T_*\mathrm{Leb} \ll \mathrm{Leb}$ (here, T_* is the pushforward of a measure and $\mathrm{Leb} = \mathrm{Leb}_{[-1,1]}$ is Lebesgue measure on $[-1, 1]$). Additionally, $\log |T'(x)| \in L^1(\mathrm{d}x)$.

(b) $(x, y) \mapsto G(x, y)$ is piecewise smooth with respect to a finite partition of $[-1, 1]^{d+1}$ into finitely many disjoint measurable sets with nonempty interior, while for each fixed $x \in [-1, 1]$ we have that $G(x, \cdot) : [-1, 1]^d \to [-1, 1]^d$ is a local diffeomorphism onto its image.

To define our perturbations, given $\omega = (\omega^1, \ldots, \omega^{d+1}) \in \mathbb{R}^{d+1}$, we set

$$F_\omega(x, y) = F(x, y) + \omega \bmod 2,$$

where the componentwise operation 'mod 2' translates a point $r \in \mathbb{R}$ to its equivalence class $r \bmod 2 \in (-1, 1]$. Given a sequence $\underline{\omega} = (\omega_1, \omega_2, \ldots)$ with $\omega_i \in \mathbb{R}^{d+1}$ for $i \geqslant 1$, we consider the random compositions

$$F^n_{\underline{\omega}} = F_{\omega_n} \circ \ldots \circ F_{\omega_2} \circ F_{\omega_1}, \quad n \geqslant 1.$$

To define the probabilistic law our perturbations ω_i will take, let $\rho : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$, what we refer to as a *mother kernel*, be a density function of bounded first variation (definition 1). Throughout, we will assume

(a) $\mathrm{Supp}\,\rho = [-1, 1]$; and

(b) There exist constants $\overline{\rho} > \underline{\rho} > 0$ such that $\underline{\rho} \leqslant \rho(x) \leqslant \overline{\rho}$ for Lebesgue a.e. $x \in [-1, 1]$.

Given $\xi \in \mathbb{R}_{>0}$, define the density

$$\rho^\xi(x) = \frac{1}{\xi}\rho\left(\frac{x}{\xi}\right).$$

When $\xi > 0$ is fixed, we assume that $\omega_1, \omega_2, \ldots$ are IID $\mathbb{R}^{d+1}$-valued random variables, each component of which is distributed like ρ^ξ. We interpret the value ξ as a *noise amplitude*, noting that $\mathrm{Supp}\,\rho^\xi = [-\xi, \xi]$ for $\xi > 0$. In the deterministic case $\xi = 0$, we follow the convention that $\omega_i \equiv 0$ for all i.

Sufficient conditions for NIO. For the fiber dynamics, we assume the following fiber-contraction property.

(F) There is a constant $c > 0$ such that $G(x, \cdot) : [-1, 1]^d \circlearrowleft$ satisfies

$$\mathrm{Lip}(G(x, \cdot)) \leqslant c < 1$$

for each fixed $x \in [-1, 1]$.

For the base dynamics, we will assume the following:

[5] It is important to note that we will allow discontinuities in $x \mapsto T(x)$ and $x \mapsto G(x, y)$ to accommodate our intended application to the contracting Lorenz map. This lack of continuity generates some issues we deal with throughout our treatment of the abstract framework.

(B)(i) (Deterministic dynamics). The deterministic base dynamics $T : [-1, 1] \circlearrowleft$ admits a unique, ergodic, absolutely continuous invariant measure μ_0 with density f_0 which is >0 almost everywhere.

Under this assumption, the Lyapunov exponent

$$\lambda_{\text{base}}(0) := \lim_n \frac{1}{n} \log |(T^n)'(x)| = \int \log |T'(x)| \mathrm{d}\mu_0(x) \tag{1}$$

exists and is x-independent for Leb. almost-every x by the Birkhoff ergodic theorem. We will additionally assume:

(B)(ii) (Positive LE). We have $\lambda_{\text{base}}(0) > 0$.

We now turn to assumptions on the base dynamics in the presence of noise. Below, given $\eta \in \mathbb{R}$ we write $T_\eta(x) = T(x) + \eta \bmod 2$. For $\xi > 0$, we let $\eta_1, \eta_2, \ldots$ be an IID sequence distributed with law ρ^ξ. Given a sequence $\underline{\eta} = (\eta_1, \eta_2, \ldots)$, we write

$$T^n_{\underline{\eta}} = T_{\eta_n} \circ \ldots \circ T_{\eta_1}.$$

(B)(iii) For every $\xi > 0$ the Markov chain

$$X_n := T_{\eta_n}(X_{n-1}) = T^n_{\underline{\eta}}(X_0)$$

on [0, 1] admits a unique stationary measure μ_ξ with density $f_\xi > 0$ Leb. almost everywhere. Moreover, this Markov chain is *exponentially mixing in* $L^1(\mathrm{d}x)$: that is, for all $\xi > 0$ there exist $C_\xi, \gamma_\xi > 0$ such that if $g_0 \in L^1(\mathrm{d}x)$ is an arbitrary density on $[-1, 1]$ and g_n denotes the density of the law of X_n where X_0 has law g_0, then

$$\|g_n - f_\xi\|_{L^1} \leqslant C_\xi \mathrm{e}^{-\gamma_\xi n}.$$

We note that under our assumptions the kernel ρ, (B)(i) implies (B)(iii); see remark 3 below.

Exponential mixing implies ergodicity[6] of μ_ξ. It follows that the Lyapunov exponent $\lambda_{\text{base}}(\xi)$ of $T^n_{\underline{\eta}}$, defined analogously to (1), exists and is almost-surely constant over typical initial conditions and with probability 1 for all $\xi > 0$. Our last assumption has to do with continuity of $\lambda_{\text{base}}(\xi)$ in ξ:

(B)(iv) The Lyapunov exponent in the base is continuous at 0 with respect to the noise size ξ, i.e., $\lambda_{\text{base}}(\xi) \to \lambda_{\text{base}}(0)$ as $\xi \to 0$.

The last assumption we make is that T is a contraction *on average* with respect to Lebesgue measure:

(C) We have

$$\int_{-1}^{1} \log |T'(x)| \mathrm{d}x < 0.$$

Note that $\lambda_{\text{base}}(0) = \int \log |T'(x)| \mathrm{d}\mu_0(x) > 0$ as in assumption (B)(ii) implies that T *expands* along μ_0-typical trajectories, while assumption (C) implies *contraction* at Lebesgue-typical points. Roughly speaking, assumptions (B) and (C) are sufficient for NIO in the base dynamic $T^n_{\underline{\eta}} : [-1, 1] \circlearrowleft$, i.e., a transition from $\lambda_{\text{base}}(\xi) > 0$ to $\lambda_{\text{base}}(\xi) < 0$ as ξ is increased [29].

[6] Recall that a stationary measure μ for a Markov chain (X_n) is *ergodic* if (X_n)-invariant sets have μ-measure 0 or 1; here, a set $A \subset [0, 1]$ is called (X_n)-invariant if with probability 1, $X_0 \in A$ if and only if $X_1 \in A$ [19].

As we show below (lemma 1), assumptions (F), (B) and (C) imply that for the deterministic map F, the Lyapunov exponent

$$\lambda(0) = \lim_n \frac{1}{n} \log \|D_{(x,y)} F^n\|$$

exists and is constant over Lebesgue-typical $(x, y) \in [-1, 1]^{d+1}$, while at positive noise $\xi > 0$, the limit

$$\lambda(\xi) = \lim_n \frac{1}{n} \log \|D_{(x,y)} F^n_{\underline{\omega}}\|$$

exists and is constant over Lebesgue-typical $(x, y) \in [-1, 1]^{d+1}$ and with probability 1 (corollary 1). Over the course of section 3, we will show the following:

(a) $\xi \mapsto \lambda(\xi)$ is continuous; [7]
(b) $\lambda(0) > 0$ (lemma 1); and
(c) $\limsup_{\xi\to\infty} \lambda(\xi) < 0$ (proposition 1 and lemma 9).

Taken together, these imply NIO in $F^n_{\underline{\omega}}$, i.e., the existence of a transition from $\lambda(\xi) > 0$ to $\lambda(\xi) < 0$:

Theorem 1 (Sufficient condition for noise-induced order). *Under assumptions (F), (B) and (C) above, there exist noise amplitudes $\xi_+ < \xi_-$ such that $\lambda(\xi) > 0$ for $\xi \in [0, \xi_+)$ and $\lambda(\xi) < 0$ for $\xi \in (\xi_-, \infty)$.*

Remark 1. As showed by numerical experiments for unimodal maps in [29] and through a rigorous computed aided proof in Lasota–Mackey maps [10] there may be more than one transition from positive to negative. Our result proves that there exists at least one such a transition.

Application to contracting Lorenz map. We will apply our results to skew-products of the form

$$F(x, y) = (T(x), G(x, y)),$$

where

$$T(x) = \begin{cases} -\alpha|x|^s + 1 & x < 0 \\ \alpha|x|^s - 1 & x > 0 \end{cases}, \quad G(x, y) = \begin{cases} -2^{-r}y|x|^r + c & x < 0 \\ 2^{-r}y|x|^r + c & x > 0 \end{cases}$$

which arise naturally as the first return maps for the contracting Lorenz flow. We will show (section 3.3) that there exist values of α and s for which the top Lyapunov exponent transitions from positive to negative as the noise size increases. As s increases the size of the contracting part of the phase space grows; a plot of the map T for parameters that present NIO can be found in figure 1.

Remark 2. We emphasize that the assumptions on the form of the noise and the mother kernel ρ are crucial to the validity of the approach in this paper: absolute continuity in the law of the ω_i (what some authors refer to as a *physical random perturbation* [28]) is used to prove almost-sure convergence of Lyapunov exponents (cf corollary 1 below), while bounded

[7] We actually only prove continuity of λ where $\lambda_{\text{base}}(\xi) > 0$, but this is sufficient for our purposes; see proposition 1 for details.

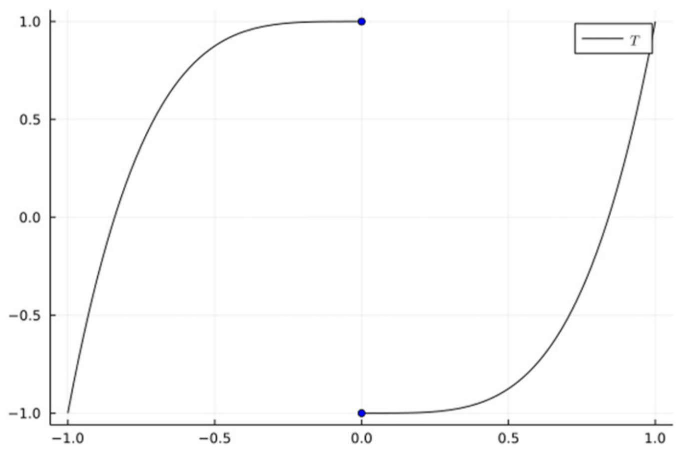

Figure 1. The map T for $s = 4$, $\alpha = 2$; the point 0 is a discontinuity point.

variation of ρ ensures equidistribution in the law of ω_i mod 2 along $[-1, 1]^{d+1}$ as $\xi \to \infty$ (lemma 9 below).

Remark 3. In our setting, ergodicity of μ_0 in assumption (B)(i) implies, by the Poincaré recurrence theorem, the existence of a dense orbit $T^n x_0$ for some $x_0 \in [-1, 1]$ such that $\{T^n x_0\}_{n\geqslant 0}$ are all continuity points for T. That (B)(iii) is satisfied for the random perturbations $T^n_{\underline{\eta}}$ now follows from [5, theorem A]. The authors gratefully thank the referees of this manuscript for making this observation.

3. Proofs

In section 3.1 we use RDS theory to establish basic properties of the Lyapunov exponents $\lambda(\xi)$, $\lambda_{\text{base}}(\xi)$ (existence, almost-sure constancy and continuity in ξ). The proof of theorem 1 is given in section 3.2, and finally, the application to contracting Lorenz maps is given in section 3.3.

3.1. Existence and properties of Lyapunov exponents

3.1.1. Deterministic case ($\xi = 0$). We begin by addressing existence of the Lyapunov exponent $\lambda(0)$ for F in the absence of noise.

Lemma 1. *The limit*

$$\lambda(0) = \lim_n \frac{1}{n} \log \|D_{(x,y)}F^n\|$$

exists and is constant ((x, y)-independent) for Leb-almost every $(x, y) \in [-1, 1]^{d+1}$, *and moreover, coincides with* $\lambda_{\text{base}}(0)$.

Proof. Let $\pi : [-1, 1]^{d+1} \to [-1, 1]$ denote projection onto the first coordinate. For skew product mappings $F : [-1, 1]^{d+1}$ satisfying the fiber contraction property (F), [6, corollary 7.22] implies there exists a unique F-invariant measure ν_0 on $[-1, 1]^{d+1}$ for which $\pi_*\nu_0 = \mu_0$. Ergodicity of μ_0 implies that of ν_0 by [6, corollary 7.25]. Consequently, the Lyapunov exponent $\lambda(0)$ exists and is constant over ν_0-typical $(x, y) \in [-1, 1]^{d+1}$ by the subadditive ergodic theorem.

On the other hand, ν_0 is singular w.r.t. Lebesgue, so an additional argument is needed to check convergence at Lebesgue-typical (x, y). We sketch this (standard) argument below, along the way introducing some notation useful later on.

Fixing x, let $D_{(x,y)}G$ denote the Jacobian of the mapping $G(x, \cdot) : [-1, 1]^d \circlearrowleft$, and fixing y, let $\nabla G(x, y) \in \mathbb{R}^d$ be the vector of partial derivatives of the components of G with respect to x. In this notation, for the full Jacobian of $F : [-1, 1]^{d+1} \circlearrowleft$ we have

$$D_{(x,y)}F = \begin{pmatrix} T'(x) & 0 \\ \nabla G(x, y) & D_{(x,y)}G \end{pmatrix}.$$

Writing $D_{(x,y)}G^n = D_{F^{n-1}(x,y)}G \circ \ldots \circ D_{(x,y)}G$, we see that

$$D_{(x,y)}F^n = \begin{pmatrix} (T^n)'(x) & 0 \\ (*) & D_{(x,y)}G^n \end{pmatrix},$$

where

$$(*) = \sum_{i=1}^{n} (T^{n-i})'(x) D_{F^{n-(i-1)}(x,y)} G^{i-1} \nabla G(F^{n-i}(x, y)). \tag{2}$$

Let $x \in [-1, 1]$ be drawn from the Leb. typical set along which $\lim_n \frac{1}{n} \log |(T^n)'(x)| = \lambda_{\text{base}}(0)$, and let $y \in [-1, 1]^d$ be arbitrary: we will show that the limit defining $\lambda(0)$ exists at all such (x, y), and coincides with $\lambda_{\text{base}}(0)$. To start,

$$\|D_{(x,y)}F^n(1, 0)\| \geqslant |(T^n)'(x)|,$$

hence $\liminf_n \frac{1}{n} \log \|D_{(x,y)}F^n\| \geqslant \lambda_{\text{base}}(0)$ for all such (x, y). For the upper bound, (2) and (F) imply that for all $\epsilon > 0$,

$$\|D_{(x,y)}F^n(1, 0)\| \leqslant C e^{n(\lambda_{\text{base}}(0)+\epsilon)},$$

where $C = C(x, \epsilon) > 0$, while for $w \in \mathbb{R}^d$, we have the bound $\|D_{(x,y)}F^n(0, w)\| \leqslant c^n$ directly from assumption (F). We conclude

$$\limsup_n \frac{1}{n} \log \|D_{(x,y)}F^n\| \leqslant \lambda_{\text{base}}(0) + \epsilon,$$

and the proof is complete on taking $\epsilon \to 0$. □

3.1.2. Preliminaries on random dynamical systems (RDS) and transfer operators. Here we will recall elements of RDS theory needed below. The Lyapunov exponent $\lambda_{\text{base}}(\xi)$ for $T^n_{\underline{\eta}} : [-1, 1] \circlearrowleft$ is considered in section 3.1.3, and finally, the exponent $\lambda(\xi)$ for the full skew product $F^n_{\underline{\omega}} : [-1, 1]^{d+1} \circlearrowleft$ is covered in section 3.1.4.

First, we briefly recall two alternative formulations of the random dynamics $(F^n_{\underline{\omega}})$ (cf [19]). The first is as a Markov chain (X_n, Y_n) on $[-1, 1]^{d+1}$ defined by

$$(X_n, Y_n) = F_{\omega_n}(X_{n-1}, Y_{n-1})$$

for fixed initial $(X_0, Y_0) \in [-1,1]^{d+1}$. Recall that the process (X_n) is a Markov chain in its own right (notation as in assumption (B)(iii)).

The second alternative formulation is as a *deterministic skew product*. For this, let $\Omega = (\mathbb{R}^{d+1})^{\otimes\mathbb{N}}$ be the sequence space of random samples $\underline{\omega} = (\omega_n)_{n\geqslant 1}$. Let $\mathcal{F}$ be the corresponding Borel σ-algebra and for $\xi > 0$, let $\mathbb{P}_\xi$ denote the probability measure on $(\Omega, \mathcal{F})$ assigning law ρ^ξ to each real coordinate. We define the (ergodic) mpt $\theta : (\Omega, \mathcal{F}, \mathbb{P}_\xi)$ to be the leftward shift, given for $\underline{\omega} = (\omega_1, \omega_2, \ldots)$ by

$$\theta\underline{\omega} = (\omega_2, \omega_3, \ldots).$$

Write $\tau :\ \Omega \times [-1,1]^{d+1} \circlearrowleft$ for the skew product system

$$\tau(\underline{\omega}; x, y) = (\theta\underline{\omega}; F_{\omega_1}(x, y)),$$

so that $\tau^n(\underline{\omega}; x, y) = (\theta^n\underline{\omega}; F^n_{\underline{\omega}}(x, y))$. Similarly, we write τ_1 for the corresponding skew product on $\Omega \times [-1,1]$ tracking only the x-coordinate; that is,

$$\tau_1(\underline{\omega}; x) = (\theta\underline{\omega}, T_{\omega_1^1}(x))$$

(recall that $\omega_i = (\omega_i^1, \ldots, \omega_i^{d+1})$ for $i \geqslant 1$).

We now turn attention to the *annealed transfer operator* associated to the base dynamics. Here, we let $T : [-1,1] \circlearrowleft$ denote any measurable, nonsingular transformation (that is, $K \subset [-1,1]$ has Lebesgue measure 0 iff $T^{-1}(K)$ has Lebesgue measure 0).

Given a measure μ on $[-1,1]$, let $T_*\mu := \mu \circ T^{-1}$ denote the *pushforward* of μ, noting that nonsingularity of T implies that if μ is absolutely continuous, then $T_*\mu$ is absolutely continuous. We define the (deterministic) *transfer operator* $\mathcal{L}_T$ to be the operator on $L^1 = L^1([-1,1])$ such that

$$\mathcal{L}_T f = \frac{\mathrm{d}T_*\mu}{\mathrm{d}x},$$

where $f \in L^1$ is arbitrary and μ is the (possibly signed) measure with $\frac{\mathrm{d}\mu}{\mathrm{d}x} = f$. In other words, the transfer operator $\mathcal{L}_T$ embodies how densities evolve under the action of the deterministic dynamics of T: if X_0 is a $[-1,1]$-valued random variable in $[-1,1]$ distributed with a density f, then $T(X_0)$ is distributed with density $\mathcal{L}_T f$.

Observe that $\|\mathcal{L}_T\|_{L^1} \leqslant 1$. Note that for T as in section 2, we have that

$$\mathcal{L}_T f(x) = \sum_{y\in T^{-1}x} \frac{f(y)}{|T'(y)|}$$

for all $f \in L^1$ and for Leb-a.e. $x \in [-1,1]$.

In the setting of section 2 and for $\xi > 0$ fixed, the *annealed* transfer operator $\mathcal{L}_\xi$ is the analogous object for the *random dynamics*. That is, if X_0 has density f, then $\mathcal{L}_\xi$ is defined to be the density of the law of $T_\eta(X_0)$, where η is distributed with density ρ^ξ. A fixed point of $\mathcal{L}_\xi$ is called a *stationary density*; the associated measure $\mathrm{d}\mu_\xi = f_\xi \mathrm{d}x$ is called a *stationary measure*.

Precisely, given $f \in L^1$, the function $\mathcal{L}_\xi f \in L^1$ is given by

$$\mathcal{L}_\xi f = \rho^\xi \hat{*} \mathcal{L}_T f,$$

where for $h:[-1,1]\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$, the periodic convolution $g\,\hat{*}\,h:[-1,1]\to\mathbb{R}$ is given[8] by the formula

$$g\,\hat{*}\,h(x)=\sum_{i\in\mathbb{Z}}\int_{-1}^{1}g(x+2i-y)h(y)\,\mathrm{d}y,$$

when the above sums and integrals exist and converge absolutely.

We close this preliminary section by recalling properties of the following natural class of densities for $\mathcal{L}_T,\mathcal{L}_\xi$.

Definition 1. Let I be an open (possibly unbounded) interval and $f:I\to\mathbb{R}$ be integrable. The *variation* $\mathrm{Var}(f)$ of f is defined by

$$\mathrm{Var}_I(f)=\sup\left\{\int f(x)\varphi'(x)\,\mathrm{d}x:\varphi\in C^1_{\mathrm{c}}(I),\|\varphi\|_\infty\leqslant 1\right\}.$$

Here, $C^1_{\mathrm{c}}(I)$ denotes the set of compactly supported C^1 functions $\varphi:I\to\mathbb{R}$, and $\|\cdot\|_\infty$ the uniform norm. We say f is of *bounded variation* if $\mathrm{Var}_I(f)<\infty$.

We observe that the set $BV(I)$ of bounded variation functions on I is a Banach space when endowed with the norm

$$\|f\|_{BV(I)}=\|f\|_{L^1(I)}+\mathrm{Var}_I(f);$$

for details, see chapter 3 of [3]. Moreover, it is straightforward to show that $BV(I)\subset L^\infty(I)$, with $\|f\|_{L^\infty(I)}\leqslant\|f\|_{BF(I)}$. We will also use below the following equivalent characterization of $\mathrm{Var}_I(f)$.

Lemma 2 (Theorem 3.27 of [3]). *Let $I=(a,b)$, $-\infty\leqslant a<b\leqslant\infty$. We have that*

$$\mathrm{Var}_I(f)=\inf_{\tilde{f}}\sup_{\mathcal{P}}\sum_{i=1}^{n-1}|\tilde{f}(x_{i+1})-\tilde{f}(x_i)|,\tag{3}$$

where $\sup_{\mathcal{P}}$ is a supremum is taken over all finite ordered tuples $a<x_1<\ldots<x_n<b$, and $\inf_{\tilde{f}}$ is an infimum taken over all $\tilde{f}\in L^1(a,b)$ such that $\tilde{f}=f$ holds Lebesgue almost-everywhere.

The following additional properties will also be used.

Lemma 3.

(*a*) *If $I\subset J$ are two open intervals and $\mathrm{Var}_J(f)<\infty$, then $\mathrm{Var}_I(f)\leqslant\mathrm{Var}_J(f)$.*

(*b*) *Let $\mathcal{I}=\{I_1,I_2,\ldots\}$ be an at-most countable open cover of an open interval I of multiplicity*[9] *at most k. Then, for any $f\in BV(I)$, we have*

$$\sum_i \mathrm{Var}_{I_i}(f)\leqslant k\,\mathrm{Var}_I(f).$$

[8] The density of the sum of two independent random variables is given by convolution of densities; the 'periodization' procedure above takes into account the fact that T_η has a 'mod 2' in its definition.

[9] Precisely, for any $x\in I$, the set $\{i:x\in I_i\}$ has cardinality $\leqslant k$.

Proof sketch. Item (a) is immediate from definition 1, while (b) follows from the characterization (3). □

The following two lemmas are vital in our proof of noise induced order in theorem 1.

Lemma 4. *Let $g \in L^1(\mathbb{R})$ be of bounded variation and $h \in L^1([-1,1])$. Then,*

$$\mathrm{Var}_{(-1,1)}(g \hat{*} h) \leqslant 2\mathrm{Var}_{\mathbb{R}}(g)\|h\|_{L^1([-1,1])}.$$

Proof. Let $\varphi \in C^1_c(-1,1)$. Then,

$$\begin{aligned}
&\int_{-1}^{1} \varphi'(x)(g\hat{*}h)(x)\mathrm{d}x \\
&\quad = \sum_{i\in\mathbb{Z}} \iint_{(-1,1)^2} \varphi'(x) g(x+2i-y)h(y)\mathrm{d}x\mathrm{d}y \\
&\quad = \sum_{i\in\mathbb{Z}} \int_{y=-1}^{1} \left(\int_{x=-1-y+2i}^{1-y+2i} \varphi'(x-2i+y)g(x)\mathrm{d}x \right) h(y)\mathrm{d}y \\
&\quad \leqslant \sum_{i\in\mathbb{Z}} \int_{y=-1}^{1} \mathrm{Var}_{(-1-y+2i,1-y+2i)}(g)|h(y)|\mathrm{d}y \\
&\quad \leqslant \|h\|_{L^1(-1,1)} \sum_{i\in\mathbb{Z}} \mathrm{Var}_{(2(i-1),2(i+1))}(g) \leqslant 2\|h\|_{L^1}\mathrm{Var}_{\mathbb{R}}(g).
\end{aligned}$$

Here, from the third to the fourth line we used lemma 3(a), while in the fourth line we used lemma 3(b) and that $\{(2(i-1), 2(i+1))\}$ covers $\mathbb{R}$ with multiplicity 2. In view of definition 1, the proof is complete. □

Lemma 5. *Let $h \in L^1(\mathbb{R})$ be of bounded variation and for $\xi > 0$ define $h^{\xi}(x) = \xi^{-1}h(x/\xi)$. Then,*

$$\mathrm{Var}_{\mathbb{R}}(h^{\xi}) = \frac{1}{\xi}\mathrm{Var}_{\mathbb{R}}(h). \tag{4}$$

Proof. Let $\varphi \in C^1_c(\mathbb{R})$ and compute:

$$\begin{aligned}
\int \varphi'(x)h^{\xi}(x)\mathrm{d}x &= \frac{1}{\xi}\int \varphi'(x)h(x/\xi)\mathrm{d}x = \int \varphi'(\xi x)h(x)\mathrm{d}x \\
&= \frac{1}{\xi}\int \psi'(x)h(x)\mathrm{d}x \leqslant \frac{1}{\xi}\mathrm{Var}_{\mathbb{R}}(h),
\end{aligned}$$

where $\psi(x) := \varphi(\xi x), \psi'(x) = \xi\varphi'(\xi x)$. Using definition 1, we conclude $\mathrm{Var}_{\mathbb{R}}(h^{\xi}) \leqslant \xi^{-1}\mathrm{Var}_{\mathbb{R}}(h)$. The opposite inequality follows similarly. □

As we commonly work with the interval $[a,b] = [-1,1]$, when there is no risk of confusion we will abuse notation and write $BV = BV([-1,1])$, $\|f\|_{BV} = \|f\|_{BV([-1,1])}$, etc.

3.1.3. Noisy base dynamics ($\xi > 0$). We now turn attention to the Lyapunov exponent in the base, $\lambda_{\mathrm{base}}(\xi)$. To start, we require the following.

Lemma 6. *Assume $\mathcal{L}_\xi$ is exponentially mixing in L^1 for all $\xi \in (0, \infty)$ in the sense of assumption (B)(iii) in section 2. For each $\xi \in (0, \infty)$, let f_ξ denote the unique stationary density, i.e., the unique density function on $[-1, 1]$ such that $\mathcal{L}_\xi(f_\xi) = f_\xi$. Then, $\xi \mapsto f_\xi$ varies continuously in the $BV(-1, 1)$ norm.*

Proof. Given a bounded operator $A : L^1 \to BV$, define

$$|||A||| := \|A\|_{L^1 \to BV} = \sup\ \{\|A\varphi\|_{BV} : \varphi \in L^1, \|\varphi\|_{L^1} \leqslant 1\}.$$

Recall that $\|\mathcal{L}_\xi\|_{L^1} = 1$, and by lemma 4, for $\varphi \in L^1$ we have

$$\mathrm{Var}(\mathcal{L}_\xi \varphi) \leqslant 2\mathrm{Var}(\rho^\xi)\|\mathcal{L}_T f\|_{L^1} \leqslant \mathrm{Var}(\rho^\xi)\|\varphi\|_{L^1},$$

hence $|||\mathcal{L}_\xi||| \leqslant \min\{1, 2\mathrm{Var}(\rho^\xi)\}$. In particular, for mean-zero $g \in L^1$ (i.e., $\int_{-1}^{1} g\mathrm{d}x = 0$), we have

$$\|\mathcal{L}_\xi^{n+1} g\|_{BV} \leqslant C_\xi \mathrm{e}^{-n\gamma_\xi} |||\mathcal{L}_\xi||| \cdot \|g\|_{L^1}.$$

Let now $\xi, \xi' > 0$ and fix n so that $C_\xi \mathrm{e}^{-n\gamma_\xi} |||\mathcal{L}_\xi||| < 1/2$. We estimate

$$\begin{aligned} \|f_\xi - f_{\xi'}\|_{BV} &= \|\mathcal{L}_\xi^{n+1} f_\xi - \mathcal{L}_{\xi'}^{n+1} f_{\xi'}\|_{BV} \\ &\leqslant \|\mathcal{L}_\xi^{n+1}(f_\xi - f_{\xi'})\|_{BV} + \|(\mathcal{L}_\xi^{n+1} - \mathcal{L}_{\xi'}^{n+1}) f_{\xi'}\|_{BV}. \end{aligned}$$

The first term is $\leqslant (1/2)\|f_\xi - f_{\xi'}\|_{L^1} \leqslant (1/2)\|f_\xi - f_{\xi'}\|_{BV}$. Using a telescoping argument and that $\mathcal{L}_{\xi'} f_{\xi'} = f_{\xi'}$, we obtain

$$\begin{aligned} \|f_\xi - f_{\xi'}\|_{BV} &\leqslant 2\|(\mathcal{L}_\xi^{n+1} - \mathcal{L}_{\xi'}^{n+1}) f_{\xi'}\|_{BV} \\ &\leqslant 2\sum_{i=0}^{n} \|\mathcal{L}_\xi^i(\mathcal{L}_\xi - \mathcal{L}_{\xi'}) f_{\xi'}\|_{BV} \\ &\leqslant 2|||\mathcal{L}_\xi - \mathcal{L}_{\xi'}||| \sum_{i=0}^{n} \|\mathcal{L}_\xi^i\|_{BV}. \end{aligned}$$

It is straightforward to check that $\mathcal{L}_\xi$ is bounded in the BV norm and $|||\mathcal{L}_\xi - \mathcal{L}_{\xi'}||| \to 0$ as $\xi' \to \xi$, completing the proof. □

The following summarizes the properties we use for the base Lyapunov exponent $\lambda_{\mathrm{base}}(\xi)$:

Lemma 7. *Assume condition (B).*

(*a*) *For all $\xi \in (0, \infty)$, the limit*

$$\lambda_{base}(\xi) = \lim_n \frac{1}{n} \log |(T^n_{\underline{\eta}})'(x)|$$

exists and is constant (independent of x) for Lebesgue-almost every $x \in [-1, 1]$ and with probability 1.

(*b*) *We have that $\xi \mapsto \lambda_{\text{base}}(\xi)$ is continuous over $\xi \in [0, \infty)$.*

Proof. For (a), the proof is to apply the Birkhoff ergodic theorem to the measure-preserving transformation $\tau_1 : \Omega \times [-1, 1] \circlearrowleft$ with invariant measure $\mathbb{P}_\xi \times \mu_\xi$, using the well-known fact that μ_ξ is an ergodic stationary measure iff $\mathbb{P}_\xi \times \mu_\xi$ is an ergodic invariant measure for τ_1 [19].

For (b), (B)(iii) ensures continuity of $\xi \mapsto \lambda_{\text{base}}(\xi)$ at $\xi = 0$. For $\xi > 0$, the Birkhoff ergodic theorem implies

$$\lambda_{\text{base}}(\xi) = \int \log |T'(x)| f_\xi(x)\mathrm{d}x \tag{5}$$

for all $\xi \in [0, \infty)$. In view of the fact that $\log |T'(x)| \in L^1(\mathrm{d}x)$, we have that

$$\left| \int \log |T'(x)| f_\xi(x)\mathrm{d}x - \int \log |T'(x)| f_{\hat{\xi}}(x)\mathrm{d}x \right| \leqslant \| \log |T'| \|_{L^1} \|f_\xi - f_{\hat{\xi}}\|_\infty.$$

Since $\|.\|_\infty \leqslant \|.\|_{BV}$, it suffices to check that $\xi \mapsto f_\xi$ varies continuously in the BV norm, as proved in lemma 6. □

3.1.4. Noisy fiber dynamics ($\xi > 0$). For the random dynamics in the full skew product, we start by checking existence and uniqueness of stationary measures ν_ξ for the Markov chain (X_n, Y_n).

Lemma 8. *Let $\xi > 0$ be arbitrary. Assume the fiber contraction condition (F) and that (X_n) has a unique, absolutely continuous, ergodic stationary measure μ_ξ (as in assumption (B)(iii)). Then, the Markov chain (X_n, Y_n) admits a unique, ergodic, absolutely continuous stationary measure ν_ξ.*

Proof. Existence follows from a mild variation of the typical Krylov–Bogoliubov argument. Given a density h on $[-1, 1]^{d+1}$, let P^*h denote the law of (X_1, Y_1) assuming (X_0, Y_0) is distributed like $h\mathrm{d}x\mathrm{d}y$. Fixing a smooth initial density h, consider the sequence

$$h_n := \frac{1}{n}\sum_{i=0}^{n-1} (P^*)^i h,$$

noting that $(P^*)^i h$ is the law of (X_i, Y_i) assuming that (X_0, Y_0) is distributed like $h\mathrm{d}x\mathrm{d}y$. By compactness of BV in L^1, there is an L^1-convergent subsequence h_{n_k} with limit $h \in L^1$. That h is an invariant density now follows from the straightforward bound $\|P^*h - P^*h_{n_k}\|_{L^1} \leqslant \|h - h_{n_k}\|_{L^1}$. Note that any stationary density h for (X_n, Y_n) is automatically BV since ρ^ξ is a BV function.

Let us now sketch the proof of unique existence using the fiber contraction property (F). Let ν_ξ be an arbitrary stationary measure for (X_n, Y_n) projecting to μ_ξ on the x-coordinate; to prove uniqueness, it suffices to show ν_ξ must be ergodic. Let $\varphi : [-1,1]^{d+1} \to \mathbb{R}$ be continuous. By the Birkhoff ergodic theorem[10] applied to $\tau : \ \Omega \times [-1,1]^{d+1} \circlearrowleft$, the limit

$$\varphi_*(\underline{\omega}; x, y) = \lim_n \frac{1}{n} \sum_0^{n-1} \varphi \circ F^n_{\underline{\omega}}(x, y)$$

exists for $\mathbb{P}_\xi \times \nu$-a.e. $(\underline{\omega}; x, y)$. To prove ν_ξ is ergodic, it suffices to show φ_* is ν_ξ-a.s. constant.

An argument using the fiber contraction property (F) implies that φ_* does not depend on y (see, e.g., section 7.3.4 of [6]), so we can view φ_* as a function only of $(\underline{\omega}; x) \in \Omega \times [-1,1]$. Since φ_* is τ_1-invariant and $\mathbb{P}_\xi \times \mu_\xi$ is ergodic for $\tau_1 : \ \Omega \times [-1,1] \circlearrowleft$ (theorem I.2.1 of [19]), we conclude φ_* is $\mathbb{P}_\xi \times \mu_\xi$-almost surely constant. Since φ was an arbitrary continuous function, uniqueness (hence ergodicity) of ν_ξ follows. □

We now turn attention to the Lyapunov exponents of $F^n_{\underline{\omega}} : [-1,1]^{d+1} \circlearrowleft$.

Corollary 1. *Assume the setting of lemma 8. For all $\xi > 0$, the limits*

$$\chi_i(\xi) = \lim_n \frac{1}{n} \log \sigma_i(D_{(x,y)} F^n_{\underline{\omega}}), \quad 1 \leqslant i \leqslant d+1$$

exist and are constant over ν_ξ-typical $(x,y) \in [-1,1]^{d+1}$ with probability 1 (some possibly equal to $-\infty$). Moreover, the limit defining

$$\lambda(\xi) := \chi_1(\xi)$$

exists and is constant over Leb*-typical $(x,y) \in [-1,1]^{d+1}$ with probability 1.*

Here, σ_i refers to the ith-largest singular value of a matrix. The values $\{\chi_i\}$ are the *Lyapunov exponents* of the derivative cocycle $D_{(x,y)} F^n_{\underline{\omega}}$. We set

$$\lambda(\xi) = \chi_1(\xi),$$

the top Lyapunov exponent.

Proof. For $\xi > 0$ and at ν_ξ-typical (x,y), everything follows from the subadditive ergodic theorem applied to the sequence of functions

$$(\underline{\omega}, (x,y)) \mapsto \|\wedge^k D_{(x,y)} F^n_{\underline{\omega}}\| = \prod_{i=1}^k \sigma_i(D_{(x,y)} F^n_{\underline{\omega}}), \quad n \geqslant 1$$

for each fixed k, viewed as subadditive over the dynamical system $\tau : \ \Omega \times [-1,1]^{d+1} \circlearrowleft$ with invariant measure $\mathbb{P}_\xi \times \nu_\xi$.

It remains to check that at $i = 1$, this convergence holds for Lebesgue-typical[11] (x,y). So as not to interrupt the flow of ideas, we carry this argument out in the appendix (section A.2). □

[10] Recall by lemma I.2.3 of [19] that ν is (X_n, Y_n)-stationary iff $\mathbb{P}_\xi \times \nu$ is invariant for $\tau : \ \Omega \times [-1,1]^{d+1} \circlearrowleft$.

[11] We note that Lebesgue-typical convergence of Lyapunov exponents was proved in the recent paper [28] for IID random compositions of smooth mappings. Unfortunately their result does not apply in our setting, since the fiber-contracting skew products F we consider here are not smooth. For the sake of completeness, we provide a full argument of Lebesgue-typical convergence in our setting.

Also of interest for us are the Lyapunov exponents in the invariant bundle $\{0\} \times \mathbb{R}^d$ tangent to the fibers. Equivalently, these are the Lyapunov exponents of the cocycle

$$B^n_{\underline{\omega};(x,y)} := D_{F^{n-1}_{\underline{\omega}}(x,y)} G \circ \cdots \circ D_{F_{\underline{\omega}}(x,y)} G \circ D_{(x,y)} G$$

on $\mathbb{R}^d$ (viewed as a cocycle over $\tau: \ \Omega \times [-1,1]^{d+1} \circlearrowleft$). The following is immediate from the subadditive ergodic theorem.

Corollary 2. *Assume the setting of lemma 8. For all $\xi > 0$, the limits*

$$\hat{\chi}_i(\xi) = \lim_n \frac{1}{n} \log \, \sigma_i(D_{(x,y)} G^n_{\underline{\omega}}), \quad 1 \leqslant i \leqslant d.$$

exist and are constant over ν_ξ-typical $(x,y) \in [-1,1]$ with probability 1 (some possibly equal to $-\infty$).

Note that the fiber contraction assumption (F) implies $\hat{\chi}_1 \leqslant \log\, c$ holds for all ξ.

3.2. Proof of theorem 1

The main step is to check the following formula for the top Lyapunov exponent $\lambda(\xi)$ of $F^n_{\underline{\omega}}$.

Proposition 1. *For all $\xi \geqslant 0$ we have that*

$$\lambda(\xi) = \max\{\lambda_{\text{base}}(\xi), \hat{\chi}_1(\xi)\}.$$

Proof. Below, we suppress ξ-dependence, writing $\chi_i = \chi_i(\xi)$, $\lambda_{\text{base}} = \lambda_{\text{base}}(\xi)$, etc.

Let $V = \{0\} \times \mathbb{R}^d$ denote the linear span of the last d coordinates in $\mathbb{R}^{d+1}$. By the skew product structure of DF, we have for $v \in V$ that

$$D_{(x,y)} F^n_{\underline{\omega}}(0, v) = B^n_{\underline{\omega};(x,y)}(v).$$

In particular,

$$\det(D_{(x,y)} F^n_{\underline{\omega}}|_V) = \det(B^n_{\underline{\omega};(x,y)}).$$

By corollary 4 (see section A.1 in the appendix) applied to $DF^n_{\underline{\omega}}$ and $B^n_{\underline{\omega};(x,y)}$, we see that there exist indices $1 \leqslant i_1 < \ldots < i_d \leqslant d+1$ such that

$$\chi_{i_1} + \cdots + \chi_{i_d} = \hat{\chi}_1 + \cdots + \hat{\chi}_d.$$

To obtain another relation, observe by the block diagonal structure of $DF^n_{\underline{\omega}}$ that

$$\det(D_{(x,y)} F^n_{\underline{\omega}}) = (T^n_{\underline{\eta}})'(x) \det(B^n_{\underline{\omega};(x,y)}),$$

where $\underline{\eta} = (\eta_1, \eta_2, \ldots)$ and $\eta_i := \omega^1_i$. We obtain immediately that

$$\chi_1 + \cdots + \chi_{d+1} = \lambda_{\text{base}} + \hat{\chi}_1 + \cdots + \hat{\chi}_d,$$

and conclude

$$\lambda_{\text{base}} = \chi_{i_*},$$

where i_* is the unique element of $\{1, \ldots, d+1\} \setminus \{i_1, \ldots, i_d\}$.

For the remaining exponents $\chi_{i_j}, 1 \leqslant j \leqslant d$, let $e_1, \ldots, e_d$ be any basis for V and let $V_j = \mathrm{Span}\{e_1, \ldots, e_j\}$, noting $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_d = V$. Iteratively applying corollary 4 we see that there is a permutation σ of $\{1, \ldots, d\}$ such that

$$\lim_n \frac{1}{n} \log |\det(D_{(x,y)} F^n_{\underline{\omega}}|_{V_j})| = \chi_{i_{\sigma(1)}} + \cdots + \chi_{i_{\sigma(j)}}$$

for each $1 \leqslant j \leqslant d$. Similarly, there is another permutation $\hat{\sigma}$ of $\{1, \ldots, d\}$ such that

$$\lim_n \frac{1}{n} \log |\det(B^n_{\underline{\omega};(x,y)}|_{V_j})| = \hat{\chi}_{\hat{\sigma}(1)} + \cdots + \hat{\chi}_{\hat{\sigma}(j)}.$$

Since $\det(D_{(x,y)} F^n_{\underline{\omega}}|_{V_j}) = \det(B^n_{\underline{\omega};(x,y)}|_{V_j})$ for all j, we conclude

$$\chi_{i_{\sigma(j)}} = \hat{\chi}_{\hat{\sigma}(j)} \quad \text{for all } 1 \leqslant j \leqslant d.$$

In summary, we have shown

- $\lambda_{\mathrm{base}} = \chi_{i_*}$ for some $i_* \in \{1, \ldots, d+1\}$.
- The list of remaining exponents $\{\chi_i : 1 \leqslant i \leqslant d+1, i \neq i_*\}$ coincides with the list $\{\hat{\chi}_i : 1 \leqslant i \leqslant d\}$, counting multiplicities.

We conclude that $\max\{\chi_i\} = \max\{\lambda_{\mathrm{base}}, \hat{\chi}_1\}$ as desired. □

We will also require the following on the behavior of $\lambda_{\mathrm{base}}(\xi)$ in the infinite-noise limit $\xi \to \infty$.

Lemma 9. *Under assumption (B)(iii), we have* $\lim_{\xi \to \infty} \lambda_{\mathrm{base}}(\xi) = \frac{1}{2} \int_{-1}^{1} \log |T'(x)| \mathrm{d}x$. *If in addition (C) holds, then* $\lim_{\xi \to \infty} \lambda_{\mathrm{base}}(\xi) < 0$

Proof. Recall that the density f_ξ of the stationary measure μ_ξ for the Markov chain (X_n) on $[-1, 1]$ satisfies

$$f_\xi = \mathcal{L}_\xi f_\xi = \rho^\xi \hat{*} \mathcal{L}_T f_\xi,$$

hence by lemmas 4 and 5,

$$\mathrm{Var}(f_\xi) = \mathrm{Var}(\rho^\xi \hat{*} \mathcal{L}_T f_\xi) \leqslant 2\, \mathrm{Var}_{\mathbb{R}}(\rho^\xi) = \frac{2}{\xi} \mathrm{Var}_{\mathbb{R}}(\rho).$$

We conclude $\|f_\xi - 1/2\|_{BV} = \mathrm{Var}(f_\xi - 1/2) = \mathrm{Var}(f_\xi) \to 0$ as $\xi \to \infty$, i.e., f_ξ converges to the constant density function 1/2. In view of the fact that $\log |T'| \in L^1(m)$ and equation (5), it follows that $\lambda_{\mathrm{base}}(\xi) \to \frac{1}{2} \int_{-1}^{1} \log |T'(x)| \mathrm{d}x$ as $\xi \to \infty$, which is <0 by assumption (C). □

We are now in position to close the proof.

Proof of Theorem 1. The ingredients we use are as follows:

(a) $\xi \mapsto \lambda_{\mathrm{base}}(\xi)$ is continuous over $\xi \in [0, \infty)$ (lemma 7(b));
(b) $\lambda_{\mathrm{base}}(0) > 0$ (assumption (B)(ii)) and $\lim_{\xi \to \infty} \lambda_{\mathrm{base}}(\xi) < 0$ (lemma 9);
(c) $\lambda(\xi) = \max\{\lambda_{\mathrm{base}}(\xi), \hat{\chi}_1(\xi)\}$ (proposition 1); and
(d) $\hat{\chi}_1(\xi) \leqslant \log c < 0$ for all ξ (assumption (F)).

It is immediate from (a) and (b) that $\exists\, 0 < \xi_+ < \xi_-$ such that $\lambda_{\text{base}}(\xi) > 0$ for $\xi \in [0, \xi_+)$ and $\lambda_{\text{base}}(\xi) < 0$ for all $\xi \in (\xi_-, \infty)$. From (c) and (d), it follows that $\lambda(\xi) > 0$ for $\xi \in [0, \xi_+)$ and $\lambda(\xi) < 0$ for $\xi \in (\xi_-, \infty)$. This completes the proof. □

3.3. Application to contracting Lorenz map

For a complete construction of the contracting Lorenz Flow we refer to [6, 14]. For the sake of this paper, what matters is that the first return map to the Poincaré section $\Sigma = ([-1, 0) \cup (0, 1]) \times [-1, 1]$, denoted by $F : \Sigma \to \Sigma$ has the form

$$F(x, y) = (T(x), G(x, y)),$$

where

$$T(x) = \begin{cases} -\alpha|x|^s + 1 & x < 0 \\ \alpha|x|^s - 1 & x > 0 \end{cases}, \qquad G(x, y) = \begin{cases} -2^{-r}y|x|^r + c & x < 0 \\ 2^{-r}y|x|^r + c & x > 0 \end{cases}$$

for some c such that the sets $F([-1, 0] \times [-1, 1])$ and $F([0, 1] \times [-1, 1])$ do not overlap and $0 < \alpha \leqslant 2$, with $r > s + 3$.

The map T satisfies the following properties:

- The order of T' at 0 is $s - 1 > 0$,
- T has a discontinuity in 0, $T(0^+) = -1$, $T(0^-) = 1$,
- $T'(x) > 0$ for $x \neq 0$,
- $\max |T'(x)|$ is attained at -1 and 1,
- T has negative Schwarzian derivative.

The parameter α can be chosen in such a way that the points 1 and -1 are preperiodic repelling (which is a Misiurewicz-type condition [22]), i.e., after a finite number of iterations the orbit of 1 (respectively -1) lands on a periodic orbit and the product of the derivative along the periodic orbit is bigger than 1. Intuitively, the main obstruction to the existence of an absolutely continuous invariant measure for the deterministic map is the fact that the contracting part of the dynamics may concentrate the measure. The hypothesis above guarantees that the measure, after being concentrated by the contracting part is spread out along the periodic orbit, guaranteeing the existence of an absolutely continuous invariant measure.

This condition is always satisfied when $\alpha = 2$ for any value of s. That both of 1, -1 are preperiodic repelling will be assumed in throughout in the following discussion.

Under these hypotheses and parameter choices it was proved in [31] that the flow associated to F admits an attractor Λ_0. Indeed, a stronger result is proved; the condition that 1 and -1 are preperiodic repelling defines a codimension two submanifold N in a C^3 neighborhood $\mathcal{U}$ of F. If we take now a C^3 curve of vector fields $\eta : (-\epsilon, \epsilon) \to \mathcal{U}$ transversal to N, such that $\eta(0) = F$, [31] proves that there exists a set of parameters E for which an attractor exists (called the Rovella parameters) and that

$$\lim_{a \to 0} \frac{m(E \cup [0, a))}{a} = 1,$$

i.e., 0 is a density point for the set of Rovella parameters. We refer to remark 5 below for an outline of the arguments that show how our result extends to the set of Rovella parameters of a perturbation.

In [24] it is proved that under the above conditions, the map T has a unique a.c.i.m. and positive Lyapunov exponent, confirming conditions (B)(i) and (B)(ii). Condition (F) is evident, and so below we carry out the remaining work of checking (B)(iii), B(iv) and (C).

3.3.1. Condition (B)(iii). It was checked directly in [25] that $\exists\ \xi_0 > 0$ such that condition (B)(iii) holds for all $\xi \in (0, \xi_0]$. It is straightforward to check that this can be promoted to all $\xi > 0$ by applying, e.g., the arguments of [29, lemma 2.3.10], where general conditions were given for deducing L^1 contraction of $\mathcal{L}_{\xi'}$ when $\mathcal{L}_\xi$ is an L^1 contraction for some $0 < \xi < \xi'$.

3.3.2. Condition (B)(iv). In view of equation (5) relating $\lambda_{\text{base}}(\xi)$ to the expectation of $\log|T'|$ with respect to $f_\xi \mathrm{d}x$, it suffices to have $f_\xi \to f_0$ in BV. In [25], it was shown that $f_\xi \to f_0$ in L^1, better known as *strong stochastic stability* of the map T—however, this is not quite enough for our purposes. Fortunately, the tower construction given in [25, 26] provides enough information, as we show below.

The tower construction consists of an extension $\hat{T} : \hat{I} \circlearrowleft$ of the dynamic T, where $\hat{I} \subset \mathbb{Z} \times [-1, 1]$ is the union of a countable collection of sets of the form $E_k := \{k\} \times B_k, k \in \mathbb{Z}$, and $\{B_k\}$ is a partition mod 0 of $[-1, 1]$, whose elements are constructed appropriately by studying the return times of the map to a neighborhood of 0.

Once $\hat{T} : \hat{I} \circlearrowleft$ is specified, it satisfies the entwining relation $\pi \circ \hat{T} = T \circ \pi$, where $\pi : \hat{I} \to [-1, 1]$ is the projection taking $(k, x) \in \{k\} \times B_k$ to its corresponding point $x \in [-1, 1]$.

In [25], a RDS comprised of compositions of the form $\hat{T}_\eta : \hat{I} \circlearrowleft, \eta \in \mathbb{R}$ was constructed analogously to the construction of the random perturbations T_η of the base map T. For each η, these random systems entwine with T_η, i.e., $\pi \circ \hat{T}_\eta = T_\eta \circ \pi$. Below, we write $\hat{m}$ to denote the natural Lebesgue measure on $\hat{I}$.

Proposition 2 ([25]). *There exists $\xi_0 > 0$ such that for all $\xi \in [0, \xi_0)$, the random system $\hat{T}_\eta$ on $\hat{I}$ admits a unique BV stationary density $\hat{f}_\xi$ with respect to Lebesgue measure $\hat{m}$ on $\hat{I}$. Moreover, these densities have the property that*

$$\hat{f}_\xi \to \hat{f}_0 \quad \text{in } BV$$

as $\xi \to 0$.

Corollary 3. *Condition (B)(iv) holds, i.e., $\lambda_{\text{base}}(\xi) \to \lambda_{\text{base}}(0)$ as $\xi \to 0$.*

Proof. By the entwining property $\pi \circ \hat{T}_\eta = T_\eta \circ \pi$, it follows that $\pi_*(\hat{f}_\xi \mathrm{d}\hat{m}) = f_\xi \mathrm{d}x$, and so

$$\begin{aligned}\lambda_{\text{base}}(\xi) &= \int_{-1}^{1} \log|T'(x)| f_\xi \mathrm{d}x = \int_{-1}^{1} \log|T'(x)| \pi_*(\hat{f}_\xi \mathrm{d}\hat{m})(x) \\ &= \sum_{k \in \mathbb{Z}} \int_{B_k} \log|T'(\pi(\hat{x}))| \hat{f}_\xi(\hat{x}) \mathrm{d}\hat{m}(\hat{x}),\end{aligned}$$

for all $\xi \in [0, \xi_0]$, ξ_0 as in proposition 2. The convergence $\lambda_{\text{base}}(\xi) \to \lambda_{\text{base}}(0)$ as in condition (B)(iii) now follows from BV convergence of $\hat{f}_\xi \to \hat{f}_0$ on exchanging the summation and limit. □

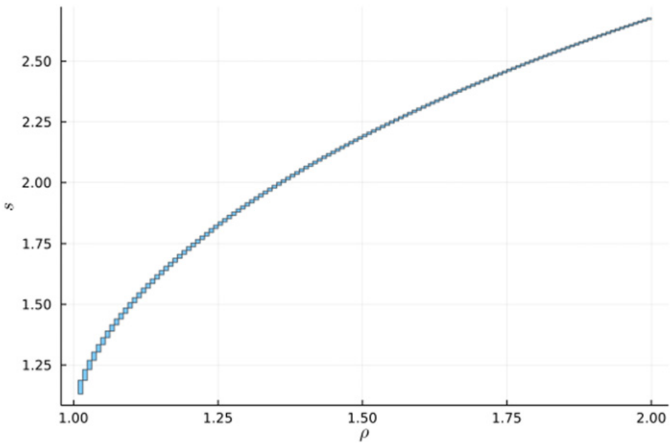

Figure 2. An enclosure for the zero set of $\lambda(\alpha, s)$, we refer to remark 4 for a formal definition

3.3.3. Condition (C).

Lemma 10. *If* $\alpha = 2$ *and* $s > 2.678\,35$ *then*

$$\frac{1}{2}\int_{-1}^{1} \log(|T'|)\mathrm{d}x < 0.$$

Proof. This follows from direct computation:

$$\lambda(\alpha, s) = \frac{1}{2}\int_{-1}^{1} \log(|T'|)\mathrm{d}x = \log(\alpha) + \log(s) + 1 - s.$$

The zero of $\lambda(2, s)$ is contained in $[2.678\,34, 2.678\,35]$; this interval is computed through the use of a rigorous interval Newton method [34]. □

Remark 4. An enclosure of the zero set of $\lambda(\alpha, s)$ for $\alpha \in [1.007\,81, 2]$ computed through the use of a rigorous interval Newton method is plotted in figure 2.

Being more precise, the plot in 2 is a family of rectangles, such that the set

$$Z = \{(\alpha, s) | \lambda(\alpha, s) = 0\}$$

is contained in the union of the rectangles.

For $s > 1$ the function $\log(s) + 1 - s$ is decreasing, therefore for each fixed α there exists a unique $s(\alpha)$ such that $\lambda(\alpha, s(\alpha)) = 0$, i.e., Z is the graph of $s(\alpha)$.

Fixed α we have that $\lambda(\alpha, s) > 0$ if $s < s(\alpha)$ and $\lambda(\alpha, s) < 0$ if $s > s(\alpha)$.

Remark 5. Arguing as in [14] it is possible to show that for C^3 perturbations of the Rovella flow associated to F associated to a Rovella parameter a, the Poincaré return map is such that

- The one dimensional map T_a is such that

$$K_2|x|^{s-1} \leqslant |T_a'(x)| \leqslant K_1|x|^{s-1}$$

for all x, a where $s = s(a) > 1$,

- T_a is C^3 and its derivative depends continuously on a.
- There exists positive constants M_1, M_2 independent of a such that

$$M_1|x|^r \leqslant |\partial_y G_a(x, y)| \leqslant M_2|x|^r.$$

This allows to extend our result to perturbations of the contracting Lorenz flow corresponding to a Rovella parameter.

4. Conclusion and outlook

Building off the previous mathematical work [13, 29] on NIO, this paper provides a rich class of higher-dimensional systems exhibiting NIO. We propose here two potential avenues for future work in this direction:

(1) Beyond skew products. It would be of considerable interest to provide examples of NIO beyond the category of skew products. A natural class potentially amenable to this kind of analysis is Henon maps and their variants at Benedicks–Carleson-type parameters, e.g.,

$$F_{a,b}(x, y) = (a + x^r + y, bx), \quad b \ll 1,$$

where $r > 2$ is a fixed parameter, so that DF is a strong contraction along 'most' of phase space, suggestive of NIO.

By now there is a well-developed stochastic stability theory for the physical measures of such systems. However, the approach to NIO presented here and in [29] requires stochastic stability not just of the physical measure but also of the *top Lyapunov exponent* of the system.

(2) Contracting Lorenz flow. Many contracting Lorenz models experience strong contraction in the vast majority of phase space, leading us to conjecture that these models also experience NIO. Deducing this from NIO for the Poincaré return map appears to be challenging, however. One concern is that noisy driving to the flow itself could destroy the Poincaré section with positive probability, making it difficult to 'lift' results for the map to the flow. For this reason, we speculate that it will be necessary to resort to computer-assisted methods such as those in [13] to deduce NIO for contracting Lorenz flow.

Acknowledgments

AB was supported by National Science Foundation Grant DMS-2009431, IN was supported by CAPES through the PROEX, and CAPES Project Nos. 88881.155748/2017-01 88887.155746/2017-00. IN would like also to thank Hokkaido University for the support and hospitality during the final revision of this article.

Appendix A

A.1. Background on the multiplicative ergodic theorem

Below, we write $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \ldots$ for the singular value of a matrix A, i.e., the eigenvalues of $\sqrt{A^\top A}$ counted with multiplicity. Recall that if A is a square $d \times d$ matrix, then $\prod_i \sigma_i(A) = |\det(A)|$.

Let $T : (X, \mathcal{F}, m) \circlearrowleft$ be an ergodic MPT of a probability space and let $A : X \to M_{d\times d}(\mathbb{R})$ be a measurable mapping. For $x \in X, n \in \mathbb{Z}_{\geqslant 1}$, let $A^n_x = A_{T^{n-1}x} \circ \ldots \circ A_x$ and assume $\log^+ \|A(x)\| = \max\{\log \|A(x)\|, 0\}$ is in $L^1(m)$. By the subadditive ergodic theorem, the limits

$$\chi_i = \lim_n \frac{1}{n} \log \sigma_i(A^n_x)$$

exist and are constant over m-typical $x \in X$ (possibly $-\infty$; here we take the convention $\log 0 = -\infty$).

Let $\lambda_1 > \lambda_2 > \ldots > \lambda_r \geqslant -\infty$ denote the distinct values among the χ_i, and let m_i denote the number of occurrences of the value λ_i (the multiplicity of λ_i). For $x \in X, v \in \mathbb{R}^d$, let

$$\lambda(x, v) = \lim_n \frac{1}{n} \log \|A^n_x(v)\|,$$

when this limit exists.

Theorem 2 (Multiplicative ergodic theorem). *At m-a.e. $x \in X$ there is a filtration*

$$\mathbb{R}^d =: F_1(x) \supsetneq F_2(x) \supsetneq \ldots \supsetneq F_r(x) \supsetneq F_{r+1}(x) := \{0\}$$

of $\mathbb{R}^d$ into measurably varying subspaces $F_i(x)$ with the property that for all $1 \leqslant i \leqslant r$ and for all $v \in F_i(x)\backslash F_{i+1}(x)$, we have

$$\lambda(x, v) = \lambda_i.$$

Below, given a square $d \times d$ matrix A and a subspace $V \subset \mathbb{R}^d$, we write $\det(A|_V)$ for the determinant of $A|_V : V \to A(V)$, using the convention $\det(A|_V) := 0$ if $\dim A(V) < \dim V$.

Corollary 4. *There is a full m-measure set of $x \in X$ for which the following holds. For any k-dimensional subspace $V \subset \mathbb{R}^d$, $1 \leqslant k \leqslant d$, there are indices $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant d$ such that*

$$\lim_n \frac{1}{n} \log |\det(A^n_x|_V)| = \chi_{i_1} + \cdots + \chi_{i_k}. \tag{6}$$

Moreover, if $V' \subsetneq V$ and $\lim_n \frac{1}{n} \log |\det(A^n_x|_{V'})| = \chi_{i'_1} + \cdots + \chi_{i'_{k'}}$, $k' = \dim V'$, then

$$\{i'_1, \ldots, i'_{k'}\} \subsetneq \{i_1, \ldots, i_k\}. \tag{7}$$

Proof sketch. Recall that a Lyapunov basis is a basis $\{v_1, \ldots, v_d\}$ of $\mathbb{R}^d$ such that $\lambda(x, v_i) = \chi_i$ for each $1 \leqslant i \leqslant d$. The proof of (6) is to construct a *Lyapunov basis* at x containing a set of k vectors which span V. To establish (7), one constructs a Lyapunov basis containing k' vectors spanning V' and k vectors spanning V; from $V' \subsetneq V$ it is immediate that the k' vectors spanning V' are contained among the k vectors spanning V. Further details are omitted. □

A.2. Completing the proof of corollary 1

We present here the argument that for Leb-almost every $(x, y) \in [-1, 1]^{d+1}$, we have that

$$\chi_1(\xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D_{(x,y)} F^n_{\underline{\omega}}\| \quad \text{with probability 1}. \tag{8}$$

A.2.1. Preliminaries. We will consider an auxiliary RDS $\hat{F}^n_{\underline{\omega}}$ on $[-1,1]^{d+1}$ obtained by first applying noise and then the map F; to wit, for $\omega \in \Omega_0$ define $\hat{F}_\omega : [-1,1]^{d+1} \circlearrowleft$ by

$$\hat{F}_\omega(x,y) = F((x,y) + \omega \mod 2),$$

and for $\underline{\omega} = (\omega_1, \omega_2, \ldots) \in \Omega, n \geqslant 1$ we set $\hat{F}_{\underline{\omega}} = \hat{F}_{\omega_n} \circ \ldots \circ \hat{F}_{\omega_1}$. The RDS $(\hat{F}^n_{\underline{\omega}})$ gives rise to a corresponding Markov chain $\{(\hat{x}_n, \hat{y}_n)\}$ on $[-1,1]^{d+1}$ defined for initial $(\hat{x}_0, \hat{y}_0) \in [-1,1]^{d+1}$ by

$$(\hat{x}_n, \hat{y}_n) = \hat{F}^n_{\underline{\omega}}(\hat{x}_0, \hat{y}_0)$$

with corresponding transition kernel $\hat{P}((x,y),K) = \mathbb{P}(\hat{F}_\omega(x,y) \in K)$.

The advantage of the auxiliary Markov chain is the following regularity property not enjoyed by the original chain (x_n, y_n).

Lemma 11. *The kernel $\hat{P}$ is strong Feller, i.e., for any bounded measurable $\varphi : [-1,1]^{d+1} \to \mathbb{R}$ we have that $\hat{P}\varphi$ is continuous.*

This is straightforward and follows from the fact that the convolution of two L^2 functions is continuous; details are omitted. By [32], it follows that $\hat{P}$ is *ultra Feller*, i.e., the transition kernels $(x,y) \mapsto \hat{P}((x,y),\cdot)$ vary continuously in the TV metric $\mathrm{dist}_{\mathrm{TV}}$, defined for Borel probability measures $\mu_1,\ \mu_2$ by

$$\mathrm{dist}_{TV}(\mu_1,\mu_2) = \frac{1}{2}\sup_A |\mu_1(A) - \mu_2(A)|.$$

We will also use the following standard point-set topology fact (proof omitted):

Lemma 12. *Let X be a compact metric space and let Y be a metric space. Let $\Phi : X \to Y$ be a continuous map. Then, Φ is uniformly continuous.*

Since $[-1,1]^{d+1}$ is compact, it follows that $(x,y) \mapsto \hat{P}((x,y),\cdot)$ is uniformly continuous in *TV*.

A.2.2. Proof of (8). Let $\hat{A} \subset [-1,1]^{d+1}$ denote the set where

$$\chi_1(\xi) = \lim_{n\to\infty} \frac{1}{n} \log \|D_{(x,y)}\hat{F}^n_{\underline{\omega}}\| \quad \text{with probability } 1,$$

and observe the following:

(a) $\hat{A}$ and the set of (x,y) where (8) holds differ on a zero Lebesgue measure set, hence it suffices to prove $\mathrm{Leb}(\hat{A}) = 1$; and

(b) To show $\mathrm{Leb}(\hat{A}) = 1$, it suffices to show that for Leb-almost every fixed initial $(\hat{x}_0, \hat{y}_0) = (x,y)$, the stopping time

$$T_{\hat{A}} = \min\{n \geqslant 1 : (\hat{x}_n, \hat{y}_n) \in \hat{A}\}$$

is almost-surely finite.

Items (a) and (b) follow from the identity

$$F^{n+1}_{\underline{\omega}}(x,y) = \omega_{n+1} + \hat{F}^n_{\underline{\omega}} \circ F(x,y)$$

and the fact that $D_{(x,y)}F$ is nonsingular almost-everywhere.

To prove $T_{\hat{A}}$ is almost-surely finite, observe that $\hat{P}((x,y),\hat{A}) = 1$ for all $(x,y) \in \hat{A}$. Using *TV* uniform continuity of $(x,y) \mapsto \hat{P}((x,y),\cdot)$, fix $\delta > 0$ so that $\hat{P}((x,y),\hat{A}) \geqslant 1/2$ whenever $\mathrm{dist}((x,y),\hat{A}) < \delta$. Now, given a fixed, Leb-typical initial $(\hat{x}_0,\hat{y}_0) = (x,y)$, there is some $y' \in [-1,1]^{d+1}$ such that $(x,y') \in \hat{A}$ (this uses that the stationary measure $\hat{\nu}_\xi$ projects to a measure $\hat{\mu}_\xi$ on the x-coordinate with density >0). Let $N \geqslant 1$ be such that $2c^N < \delta$, where $c \in (0,1)$ is as in condition (F), and observe that for our Lebesgue-typical (x,y), we have that

$$\mathrm{dist}(\hat{F}^n_{\underline{\omega}}(x,y),\hat{A}) \leqslant |\hat{F}^n_{\underline{\omega}}(x,y) - \hat{F}^n_{\underline{\omega}}(x,y')| \leqslant 2c^n < \delta$$

for all $n \geqslant N$.

We now check by induction that

$$\mathbb{P}((\hat{x}_{N+j},\hat{y}_{N+j}) \notin \hat{A}, 1 \leqslant j \leqslant i) \leqslant 2^{-i} \tag{9}$$

for all $i \geqslant 1$. Assuming this, we immediately obtain $\mathbb{P}(T_{\hat{A}} > N+i) \leqslant 2^{-i}$ which implies $T_{\hat{A}} < \infty$ with probability 1.

To prove (9): in the case $i = 1$ we have

$$\begin{aligned}\mathbb{P}((\hat{x}_{N+1},\hat{y}_{N+1}) \notin \hat{A}) &\leqslant \mathbb{P}(\hat{F}^{N+1}_{\underline{\omega}}(x,y) \notin \hat{A}) \\ &= \int \hat{P}^N((x,y),d(\hat{x},\hat{y}))\hat{P}((\hat{x},\hat{y}),\hat{A}^c) \leqslant \frac{1}{2}\end{aligned}$$

since $\hat{P}^n((x,y),\cdot)$ as a measure assigns full probability to the set where $\mathrm{dist}((\hat{x},\hat{y}),\hat{A}) < \delta$ for all $n \geqslant N$. Assuming $\mathbb{P}(T_{\hat{A}} > N+i) \leqslant 2^{-i}$, we now have

$$\begin{aligned}&\mathbb{P}((\hat{x}_{N+j},\hat{y}_{N+j}) \notin \hat{A}, 1 \leqslant j \leqslant i+1) \\ &\quad = \mathbb{P}((\hat{x}_{N+i+1},\hat{y}_{N+i+1}) \notin \hat{A} | (\hat{x}_{N+j},\hat{y}_{N+j}) \notin \hat{A}, 1 \leqslant j \leqslant i) \\ &\qquad \times \mathbb{P}((\hat{x}_{N+j},\hat{y}_{N+j}) \notin \hat{A}, 1 \leqslant j \leqslant i) \\ &\quad = \mathbb{P}((\hat{x}_{N+i+1},\hat{y}_{N+i+1}) \notin \hat{A} | (\hat{x}_{N+i},\hat{y}_{N+i}) \notin \hat{A}) \times 2^{-i},\end{aligned}$$

on combining the induction hypothesis and the Markov property in the last line. Now,

$$\begin{aligned}&\mathbb{P}((\hat{x}_{N+i+1},\hat{y}_{N+i+1}) \notin \hat{A} | (\hat{x}_{N+i},\hat{y}_{N+i}) \notin \hat{A}) \\ &\quad = \frac{1}{\hat{P}^{N+i}((x,y),\hat{A}^c)} \int_{(\hat{x},\hat{y})\notin\hat{A}} \hat{P}^{N+i}((x,y),d(\hat{x},\hat{y}))\hat{P}((\hat{x},\hat{y}),\hat{A}^c) \\ &\quad \leqslant 1/2,\end{aligned}$$

completing the proof of (9). □

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