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Rigorous Computation of Linear Response for Intermittent Maps

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Abstract
We present a rigorous numerical scheme for the approximation of the linear response of the invariant density of a map with an indifferent fixed point with respect to the order of the fixed point, with explicit and computed estimates for the error and all the involved constants.

Keywords Linear response · Intermittent maps · Transfer operators · Rigorous approximations

Mathematics Subject Classification Primary 37A05 · 37E05

1 Introduction

In [43] Ruelle proved that for certain perturbations of uniformly hyperbolic deterministic dynamical systems the underlying SRB measure changes smoothly. He also obtained a formula for the derivative of the SRB measure, called the *linear response formula* [43].[1] Since then, the topic of linear response has been a very active direction of research in smooth ergodic theory. Indeed, the work of Ruelle was refined in the uniformly hyperbolic setting [12, 28], extended to the partially hyperbolic setting [18], and has been a topic of deep investigation for unimodal maps, see [8], the survey article [9], the recent works [3, 10, 17, 44] and references therein. More recently, the topic of linear response was also studied in the

[1] See also earlier related work [33]. See also [27] for a comprehensive historical account including literature from physics.

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context of random or extended systems [7, 19, 21, 24, 35, 45, 50]. Optimisation of statistical properties through linear response was developed in [1, 2, 23, 34].

Numerical algorithms for the approximation of linear response for uniformly expanding maps, via finite rank transfer operators were obtained in [6] and via dynamical determinants and periodic orbits in [42],[2] and for uniformly hyperbolic systems in [29, 39, 40].

We wish to study the linear response of intermittent maps. Numerical work has also been done for the invariant densities of intermittent maps in [5] where an Ulam method is used, and in [49] where very high precision is reached by the use of Abel functions. Our work extends the methods in [6] to intermittent maps away from the boundary, and possibly really far from the boundary with sufficient computing power, allowing us to compute the linear response for LSV maps, a version of the Manneville-Pomeau family [38], as the exponent at the indifferent fixed point changes.

Our algorithm permits rigorous control of all the numerical and discretization errors; some non rigorous numerical schemes have been proposed in the applied community for the computation of response in the uniformly hyperbolic setting [14, 15, 40, 41], and rely on the hyperbolicity of the system for the proof of their convergence; we do not know if they can be extended to the intermittent case, due to the singular behaviour of the invariant density and their non-uniform hyperbolicity but our result may be used as a reliable benchmark.

Linear response for indifferent fixed point maps has been investigated in [4, 11, 36], but three important questions have to be addressed to obtain a rigorous numerical approximation scheme:

(1) how to approximate efficiently the involved discretized operators;
(2) how to bound the approximation errors involved in the discretization;
(3) how to bound explicitly and efficiently the constants used in the proofs of [4, 36].

In our paper we provide answers to the three questions above for general intermittent maps and present an explicit computation for LSV type maps, obtaining the following result.

Theorem 1.1 (Main numerical result) *Let*

$$T_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) \; if\, x \in [0, \frac{1}{2}] \\ 2x-1 \; if\, x \in (\frac{1}{2}, 1] \end{cases} \quad ;$$

let $\alpha_0 = 1/8$ *and for* $\epsilon \in (-\epsilon, \epsilon)$ *denote by* h_ϵ *the a.c.i.p. of* $T_{\alpha_0+\epsilon}$.

We can explicitly compute a piecewise constant function

$$h^*_\eta(x) = \sum_{i=0}^{1/\eta} v_i \chi_{[i\eta,(i+1)\eta)}(x)$$

such that

$$\lim_{\epsilon\to 0} \left\| \frac{h_\epsilon - h_0}{\epsilon} - h^*_\eta \right\|_1 \leq 0.01501$$

Moreover, the algorithm and estimates developed work for any $\alpha_0 \in (0, 1)$ *and allow the error to be made as small as wanted, theoretically.*

The first question is addressed in two steps, firstly in Sect. 3.2, with the induced map in mind, we first bound the distance with respect to adequate norms, between the transfer

[2] See also [32] for related work on dynamical determinants.

operator of a map with infinite branches and an approximating map with finitely many branches.

Then we use a matrix defined by a projection, onto a finite dimensional space, so that the invariant density is close to that of the original map's. For calculating the invariant density for the induced map, in Sect. 3.3, a Chebyshev interpolation is shown to give us a good approximation, however for calculating the linear response in Sect. 4 we require at some point an approximation of an integrable unbounded function, for which the polynomial Chebyshev interpolation is not the best tool, and for which we use an Ulam like approximation.

For the second question, error of the finite branch approximation is addressed in Lemma 3.3, while the error from the Chebyshev interpolation is addressed in theorems 3.12, 3.13 and 3.14. The error in the Ulam like discretisation is addressed in Lemma 4.2.

The final question is addressed in Appendix.

Our scheme and techniques are very flexible and can be easily adapted to other one dimensional non-uniformly expanding maps whose associated transfer operators do not admit a spectral gap (or a uniform spectral gap) as long as the linear response formula can be obtained via inducing with the first return map and the return times are summable.

In the text are presented some numerical remarks, that allow the reader to get an overview of some of the delicate points of the implementation; we made a strong effort in allowing readability of the code, with comment and citations in the source to the relevant theoretical background.

1.1 Plan of the Paper

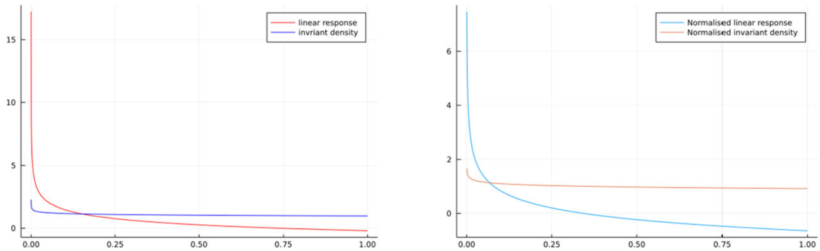
The paper is divided as follows: in Sect. 2 we state the hypothesis on the dynamical system and state our results, in Sect. 3 we present the theory behind the approximation of the density for the induced map, in Sect. 4 we discuss the approximation of the linear response for the induced map, in Sect. 5 we discuss pulling back the measure to the original map and normalizing the density, in Sect. 6 we give a proof of the fact that the error may be made as small as wanted, in Sect. 7 we compute an approximation with an explicit error of the linear response for an LSV map; Sect. 8 is devoted to computing effective bounds for the constants in [4, 36] and Appendix A explains the technique we use to compute some of the functions involved in our approximation.

2 Hypothesis on the Map and Statement of the Results

We are interested in approximating the invariant density and linear response for one dimensional interval maps with an indeterminate fixed point by inducing as the order of the fixed point changes.

In particular we wish to gain explicitly calculable error bounds in the L^1 norm. We use an induced map on [0.5, 1], to gain a map with good statistical properties to approximate an invariant density and linear response, and then using a formula used in [4] to pull back our approximation to the invariant density and linear response of the full map. We apply this method to a family of Pomeau-Manneville maps to gain an approximation of the statistics with explicit error as can be seen in figure 1.

In this section we focus on the requirements on the dynamic that make the existence and estimation of these statistical values possible and where appropriate the bounds from our requirements can be found in Appendix for our example map.



(A) The linear response and invariant density of the LSV map for $\alpha = 0.125$, calculated according to section 2.1 with L^1 error for the linear response of 0.007765.

(B) This shows the normalised invariant density of the LSV map at $\alpha = 0.125$ with the linear response of the normalised density.

Fig. 1 Here we present the invariant density of the Pomeau-Manneville map with parameter $\alpha = 0.125$. Once with an unnormalised invariant density, as we calculate it, and the corresponding linear response, and once where we impose that the invariant density should be normalised to an integral of 1 for all parameters near to, and including, α

2.1 Inducing Maps with an Indifferent Fixed Point

Here we define a family of non-uniformly expanding maps with an indifferent fixed point for which we define a further family of induced maps. Here we focus on LSV maps however a similar scheme for maps with more branches can be constructed along the same lines.

Definition 2.1 In the following, we will denote by $(\mathcal{B}, \|.\|_\gamma)$ the Banach space defined as

$$\mathcal{B} = \{f \in C^0(0, 1] \mid \|f\|_\gamma < +\infty\}$$

where $\|f\|_\gamma = \sup_{x\in(0,1]} |x^\gamma f(x)|$ and $\gamma > 0$ is fixed.

Definition 2.2 We say a map $S : [0, 1] \to [0, 1]$ is a **LSV** map if

(1) S is nonsingular with respect to Lebesgue measure m,
(2) S has two onto branches, $S_0 : [0, 0.5) \to [0, 1)$, and $S_1 : [0.5, 1] \to [0, 1]$
(3) S_0 and S_1 have inverses g_0 and g_1 respectively, and $g_i \in C^3$ for $i = 0, 1$,

Definition 2.3 Let V be a neighbourhood of 0, we say that a family of dynamical systems depending on the parameter $\epsilon \in V$ is a **family of LSV maps** if for any $\epsilon \in V$, T_ϵ is a LSV map.

Fixing notation, for each ϵ we will denote by $T_{0,\epsilon}$ the first branch of T_ϵ, by $g_{0,\epsilon}$ its inverse and follow the same convention for $T_{1,\epsilon}$, $g_{1,\epsilon}$.

We call T_0 the **unperturbed** system.

Definition 2.4 Let V be a neighbourhood of 0, we say that a family of LSV maps **inherits the linear response from the induced system** if the following are satisfied:

(1) for each $i = 0, 1$ and $j = 0, 1, 2$ the following partial derivatives exist and satisfy the commutation relation

$$\partial_\epsilon g_{i,\epsilon}^{(j)} = (\partial_\epsilon g_{i,\epsilon})^{(j)}. \tag{2.1}$$

(2) We assume that T_ϵ has a unique absolutely continuous invariant measure (acim) with a density with respect to the Lebesgue measure denoted by h_ϵ.

(3) Let Ω be the set of finite sequences of the form $\omega = 10^n$, for $n \in \mathbb{N} \cup \{0\}$. We set $g_{\omega,\epsilon} = g_{1,\epsilon} \circ g^n_{0,\epsilon}$. Then for $x \in [0, 1]$ we have $T^{n+1}_\epsilon \circ g_{\omega,\epsilon}(x) = x$. For all $\epsilon \in V$ we require the sets $g_{\omega,\epsilon}(\Delta)$ to form a partition of Δ up to a set of Lebesgue measure 0.

(4) For $x \in [0, 1]$ the following regularity assumptions hold,

$$\sup_{\epsilon \in V} \sup_{x \in [0,1]} |g'_{\omega,\epsilon}(x)| < \infty; \tag{2.2}$$

$$\sup_{\epsilon \in V} \sup_{x \in [0,1]} |\partial_\epsilon g_{\omega,\epsilon}(x)| < \infty; \tag{2.3}$$

$$\sum_\omega \sup_{\epsilon \in V} \|g'_{\omega,\epsilon}\|_{\mathcal{B}} < \infty; \tag{2.4}$$

and

$$\sum_\omega \sup_{\epsilon \in V} \|\partial_\epsilon g'_{\omega,\epsilon}\|_{\mathcal{B}} < \infty, \tag{2.5}$$

Remark 2.5 Assumption (1) guarantees that we can express the linear response formula (2.10) in terms of the spatial derivative, assumptions (2) is self explanatory, and assumption (3) guarantees a well behaved induced map. The assumptions of (4) are necessary for the proof of Theorem 2.18 i.e. they allow us to prove that for each $\epsilon \in V$ there exists a unique invariant density, and that we have some uniform bounds on the regularity of the density with respect to ϵ.

Remark 2.6 We remark that Item (1) contains a regularity assumption on the dynamic.

Remark 2.7 The LSV map for $\alpha < 1$ satisfies the hypothesis above, please note that they do not work in the infinite measure case.

Definition 2.8 For each ϵ let $\hat{T}_\epsilon$, be the first return map of T_ϵ to Δ, where $\Delta := [0.5, 1]$; i.e., for $x \in \Delta$

$$\hat{T}_\epsilon(x) = T_\epsilon^{R_\epsilon(x)}(x),$$

where

$$R_\epsilon(x) = \inf\{n \geq 1 : T^n_\epsilon(x) \in \Delta\}.$$

Definition 2.9 We denote by L_ϵ the **Perron-Frobenius operator** associated with T_ϵ; i.e., for $\varphi \in L^\infty$ and $\psi \in L^1$

$$\int \varphi \circ T_\epsilon \cdot \psi dm = \int \varphi \cdot L_\epsilon \psi dm.$$

2.2 Expanding Induced Maps

Here we discuss a family of induced maps of the sort defined by the first return map discussed in the previous section. The following follow from the assumptions on the original family of $\{T_\epsilon\}_{\epsilon \in V}$.

Lemma 2.10 ([4], Properties of the induced map) *For each $\epsilon \in V$ the induced map $\hat{T}_\epsilon : \Delta \to \Delta$ satisfies:*

(1) *the restriction of $\hat{T}_\epsilon$ to $\Delta_{\omega,\epsilon}$ is piecewise C^3, onto and uniformly expanding in the sense that* $\inf_\omega \inf_{\Delta_{\omega,\epsilon}} |\hat{T}'_{\omega,\epsilon}| > 1$

(2) *for each* $\omega \in \Omega$ *and* $j = 0, 1, 2$ *the following partial derivatives exist and satisfy the commutation relation*

$$\partial_\epsilon g^{(j)}_{\omega,\epsilon} = (\partial_\epsilon g_{\omega,\epsilon})^{(j)}.$$

(3)
$$\sup_{\omega} \sup_{\epsilon \in V} \sup_{x \in \Delta} \left| \frac{g''_{\omega,\epsilon}(x)}{g'_{\omega,\epsilon}(x)} \right| < \infty;$$

(4) *for* $i = 2, 3$

$$\sum_{\omega} \sup_{\epsilon \in V} \sup_{x \in \Delta} |g^{(i)}_{\omega,\epsilon}(x)| < \infty;$$

(5) *for* $i = 1, 2$

$$\sum_{\omega} \sup_{\epsilon \in V} \sup_{x \in \Delta} |\partial_\epsilon g^{i}_{\omega,\epsilon}(x)| < \infty.$$

Remark 2.11 Property (1) guarantees that there exists a spectral gap and a unique invariant density for each ϵ in V, (2) is self explanatory, (3) is necessary for Lemma 3.3, and (4) and (5) are necessary for the convergence of formulas (2.6) and (2.9).

Lemma 2.12 ([4], Properties of the transfer operator of the induced map) *Let* $\hat{L}_\epsilon$ *denote the transfer operator of the map* $\hat{T}_\epsilon$*; i.e., for* $\Phi \in L^1(\Delta)$

$$\hat{L}_\epsilon \Phi(x) := \sum_{\omega \in \Omega} \Phi \circ g_{\omega,\epsilon}(x) g'_{\omega,\epsilon}(x)$$

for a.e. $x \in \Delta$.

Then, $\hat{L}_\epsilon$ *has a spectral gap when acting on* C^k *and* $W^{k,1}$*,* $k = 1, 2$*, and has a unique (up to multiplication) fixed point, denoted by* $\hat{h}_\epsilon$*, i.e.,* $\hat{T}_\epsilon$ *admits a unique finite absolutely continuous invariant measure* $\mu_\epsilon = \hat{h}_\epsilon dm$.

2.3 Relating the Properties of the Induced Map and of the Original Map

Definition 2.13 We define the **pull-back operator** F as follow: For $\Phi \in L^1$, let

$$F_\epsilon(\Phi)(x) := 1_\Delta \Phi(x) + (1 - 1_\Delta) \sum_{\omega \in \Omega} \Phi \circ g_{\omega,\epsilon}(x) \cdot g'_{\omega,\epsilon}(x). \tag{2.6}$$

Remark 2.14 Note that F_ϵ is a linear operator. In fact, for $x \in [0, 1] \setminus \Delta$, the formula of F_ϵ can be re-written using the Perron-Frobenius operator of T_ϵ:

$$F_\epsilon(\Phi) := 1_\Delta \Phi + (1 - 1_\Delta) \sum_{k \geq 1} L^k_\epsilon(\Phi \cdot 1_{\{R_\epsilon > k\}}), \tag{2.7}$$

where L_ϵ is the Perron-Frobenius operator of T_ϵ.

Lemma 2.15 ([4]) *The invariant densities of the original system and the induced one are related (modulo normalization in the finite measure case) by*

$$h_\epsilon = F_\epsilon(\hat{h}_\epsilon). \tag{2.8}$$

Definition 2.16 We define the **derivative of the pull-back operator**, which represents $\partial_\epsilon F_\epsilon \Phi|_{\epsilon=0}$

$$Q\Phi(x) = (1 - 1_\Delta) \sum_\omega \Phi' \circ g_\omega(x) \cdot a_\omega(x) g'_\omega(x) + \Phi \circ g_\omega(x) \cdot b_\omega(x), \tag{2.9}$$

where $a_\omega = \partial_\epsilon g_{\omega,\epsilon}|_{\epsilon=0}$ and $b_\omega = \partial_\epsilon g'_{\omega,\epsilon}|_{\epsilon=0}$.

Lemma 2.17 ([4], Linear response formula) *The invariant density $\hat{h}_\epsilon$ of the induced map $\hat{T}_\epsilon$ is differentiable as a C^0 element and its linear response formula is given by*

$$\hat{h}^* := (I - \hat{L})^{-1} \hat{L}[A_0 \hat{h}' + B_0 \hat{h}], \tag{2.10}$$

where $\hat{h}'$ is the spatial derivative of $\hat{h}$ and

$$A_0 = -\left(\frac{\partial_\epsilon \hat{T}_\epsilon}{\hat{T}'_\epsilon} \right)\Big|_{\epsilon=0}, \qquad B_0 = \left(\frac{\partial_\epsilon \hat{T}_\epsilon \cdot \hat{T}''_\epsilon}{\hat{T}'^2_\epsilon} - \frac{\partial_\epsilon \hat{T}'_\epsilon}{\hat{T}'_\epsilon} \right)\Big|_{\epsilon=0}.$$

Moreover, for the original map, $\epsilon \mapsto h_\epsilon$ is differentiable as an element of $\mathcal{B}$; in particular, if the conditions hold for some $\gamma < 1$

$$\lim_{\epsilon \to 0} \left\| \frac{h_\epsilon - h}{\epsilon} - h^* \right\|_1 = 0,$$

and h^* is given by [3]

$$h^* = F_0 \hat{h}^* + Q\hat{h}. \tag{2.11}$$

2.4 Main Result and Explicit Strategy

We focus on the case $\gamma < 1$. The goal of this work is to provide a numerical scheme that can rigorously approximate h^*, up to a pre-specified error $\tau > 0$, in the L^1-norm. To obtain such a result we follow the following steps:

(1) first provide a sequence of finite rank operators $\hat{L}_\eta$ that can be used to approximate the linear response for the induced map $\hat{h}^*$ in $C^1(\Delta)$. Since the formula of $\hat{h}^*$ involves $\hat{h}$ and $\hat{h}'$, we will design $\hat{L}_n$ so that its invariant density, $\hat{h}_n$, well approximates $\hat{h}$ in the C^1-norm,

(2) we pull-back to the original map by defining F_0^{app} and Q_0^{app} by truncating (2.6) and (2.9); i.e., for $\Phi \in L^1$,

$$F_0^{\text{app}}(\Phi) := 1_\Delta \Phi + (1 - 1_\Delta) \sum_{\omega=1}^{N^*} \Phi \circ g_{\omega,0} g'_{\omega,0}$$

and

$$Q^{\text{app}} \Phi = (1 - 1_\Delta) \sum_{\omega=1}^{N^*} \Phi' \circ g_\omega \cdot a_\omega g'_\omega + \Phi \circ g_\omega \cdot b_\omega$$

[3] Note that in the finite measure case, h^* is the derivative of the non-normalized density h_ϵ. The advantage in working with h_ϵ is reflected in keeping the operator F_ϵ linear and to accommodate the infinite measure preserving case. In the finite measure case, once the derivative of h_ϵ is obtained, the derivative of the normalized density can be easily computed. Indeed, $h_\epsilon = h + \epsilon h^* + o(\epsilon)$. Consequently, $\int h_\epsilon = \int h + \epsilon \int h^* + o(\epsilon)$. Hence, $\partial_\epsilon (\frac{h_\epsilon}{\int h_\epsilon})|_{\epsilon=0} = h^* - h \int h^*$.

(3) finally, find N^* large enough and set

$$h_\eta^* := F_0^{\text{app}} \hat{h}_u^* + Q^{\text{app}} \hat{h}_n \tag{2.12}$$

so that

$$\|h_\eta^* - h^*\|_1 \leq \tau.$$

This strategy allows us to prove the following theorem.

Theorem 2.18 *For any $\tau > 0$, there exists:*

(1) *a sequence of finite rank operators $\hat{L}_\eta : C^1(\Delta) \to C^1(\Delta)$,*
(2) *a sequence of finite rank operators F_0^{app},*
(3) *a sequence of finite rank operators Q^{app},*
(4) *η small enough,*
(5) *$N^* > 0$ large enough,*

such that

$$\|h_\eta^* - h^*\|_1 \leq \tau.$$

2.5 The Validated Numerics Toolbox

While the strategy for the approximation of the linear response may seem quite simple, to make it rigorous, i.e., with a certified control on the error terms so that the results have the strength of proofs, many different quantities have to be estimated explicitly by means of a priori and a posteriori estimates.

The main toolbox we use for these validated estimates consists of

(1) Interval Arithmetics and rigorous contractors as the Interval Newton Method and the Shooting Method [47]
(2) discretization of the transfer operator, using the Ulam and Chebyshev basis [22, 26, 48]
(3) a priori estimate on the tail of a series and rigorous bounds for a finite number of terms.

We will introduce these methods and some of their implementation details during the proof of our result, showing the difference with the cited references when needed.

3 Approximating the Invariant Density of the Induced Map

To approximate the invariant density for the induced map two approximation steps are needed. First we need to approximate the induced map, which has countable branches with a map with a finite number of branches. Then, we will discretize the transfer operator of this map by using a Chebyshev approximation scheme.

3.1 The Involved Function Spaces

During the course of the paper, in addition to the space $\mathcal{B}$, we will use the function spaces $C^k(I)$ and $W^{k,p}(I)$, defined as follow.

Definition 3.1 Given an interval I we define the space $C^k(I)$ as the space of the k-continuously differentiable functions, endowed with the norm

$$\|f\|_{C^k} = \sum_{i=0}^{k} \|f^{(i)}\|_\infty.$$

Definition 3.2 Given an interval I we define the space $W^{k,p}(I)$ as the space of functions $f \in L^p$ with k weak derivatives $f^{(i)}$ such that

$$\|f\|_{W^{k,p}} = \sum_{i=0}^{k} \|f^{(i)}\|_{L^p} < +\infty.$$

3.2 From Countable Branches to Finite Branches

Let $\delta_k > 0$ with $\delta_k = \left|\cup_{n=k}^{\infty}[\omega]\right|$. To simplify notation we assume without loss of generality that $\frac{1}{2} \in \overline{\cup_{n=k}^{\infty}[\omega]}$. Let

$$\hat{T}_{\delta_k}(x) = \begin{cases} \hat{T}(x) & \text{, if } x \in [\delta_k, 1], \\ \frac{1}{2}\delta_k^{-1}(x - \frac{1}{2}) + \frac{1}{2} & \text{, if } x \in [0.5, \delta_k). \end{cases}$$

Then the transfer operator $\hat{L}_{\delta_k}$, associated with T_{δ_k} is acting on $\Phi \in L^1(\Delta)$ as:

$$\hat{L}_{\delta_k}\Phi(x) := \sum_{\substack{\omega\in\Omega \\ n<k}} \Phi \circ g_{\omega,\epsilon}(x) g'_{\omega,\epsilon}(x) + \Phi\left(\delta_k(2x-1) + \frac{1}{2}\right) 2\delta_k$$

for a.e. $x \in \Delta$.

Lemma 3.3 *Let $\Phi \in C^1$, then*

$$\|(\hat{L} - \hat{L}_{\delta_k})\Phi\|_{C^1} \le (D + D_0 D + 2)\|\Phi\|_{C^1}\delta_k,$$

where $D_0 = \left\| \frac{g''_\omega}{g'_\omega} \right\|_\infty$ and $D \ge 2\sup_\omega \frac{|g'_\omega(x)|}{|g'_\omega(y)|}$ for all[4] *x, y in* $[0.5, 1]$.

Proof First notice that

$$\begin{aligned} \left|(\hat{L} - \hat{L}_{\delta_k})\Phi\right| &= \left| \sum_{\substack{\omega\in\Omega \\ n\ge k}} \Phi \circ g_\omega(x) g'_\omega(x) - \Phi(\delta_k(2x-1) + \frac{1}{2})2\delta_k \right| \\ &\le \sum_{\substack{\omega\in\Omega \\ n\ge k}} |\Phi \circ g_\omega(x)| \cdot |g'_\omega(x)| + |\Phi(\delta_k(2x-1) + \frac{1}{2})|2\delta_k \end{aligned} \tag{3.1}$$

[4] The existence of a uniform constant $D > 0$ is implied by condition (3).

and

$$\begin{aligned}
&\left|\left((\hat{L}-\hat{L}_{\delta_k})\Phi\right)'\right| \\
&= \left|\sum_{\substack{\omega\in\Omega\\ n\geq k}} \left(\Phi'\circ g_\omega(x)(g'_\omega(x))^2 + \Phi\circ g_\omega(x) g''_\omega(x)\right) - |\Phi'(\delta_k(2x-1)+\frac{1}{2})4\delta_k^2|\right| \\
&\leq \sum_{\substack{\omega\in\Omega\\ n\geq k}} |\Phi'\circ g_\omega(x)|\cdot(g'_\omega(x))^2 + \sup_\omega \left\|\frac{g''_\omega}{g'_\omega}\right\|_\infty \sum_{\substack{\omega\in\Omega\\ n\geq k}} |\Phi\circ g_\omega(x)|\cdot|g'_\omega(x)| \\
&\quad + |\Phi'(\delta_k(2x-1)+\frac{1}{2})|4\delta_k^2.
\end{aligned} \tag{3.2}$$

Now notice that by the Mean Value Theorem, $\exists\, \xi_\omega \in (\frac{1}{2}, 1)$ such that

$$|g_\omega(1) - g_\omega\left(\frac{1}{2}\right)| = |g'_\omega(\xi_\omega)|/2.$$

Therefore,

$$|g'_\omega(x)| \leq 2|g_\omega(1) - g_\omega\left(\frac{1}{2}\right)| \cdot \sup_\omega \frac{|g'_\omega(x)|}{|g'_\omega(\xi_\omega)|} := D\cdot|g_\omega(1) - g_\omega\left(\frac{1}{2}\right)|. \tag{3.3}$$

Thus, using (3.3) in (3.1) and (3.2), we obtain

$$\begin{aligned}
\left|(\hat{L}-\hat{L}_{\delta_k})\Phi\right| + \left|\left((\hat{L}-\hat{L}_{\delta_k})\Phi\right)'\right| &\leq \|\Phi\|_{C^0} D \sum_{\substack{\omega\in\Omega\\ n\geq k}} |g_\omega(1) - g_\omega(\frac{1}{2})| + \|\Phi\|_{C^0} 2\delta_k \\
&+ \left(\|\Phi'\|_{C^0} D + \|\Phi\|_{C^0} D \sup_\omega \left\|\frac{g''_\omega}{g'_\omega}\right\|_\infty\right) \sum_{\substack{\omega\in\Omega\\ n\geq k}} \cdot|g_\omega(1) - g_\omega(\frac{1}{2})| + \|\Phi'\|_{C^0} 4\delta_k^2 \\
&= (D + D_0 D + 2)\|\Phi\|_{C^0}\delta_k + \|\Phi'\|_{C^0} 4\delta_k^2 \leq (D + D_0 D + 2)\|\Phi\|_{C^1}\delta_k.
\end{aligned} \tag{3.4}$$

□

The next lemma shows that using the above information, the densities $\hat{h}$ and $\hat{h}_{\delta_k}$ can be made arbitrarily close in C^1.

Lemma 3.4 *Given two operators, L_1 and L_2, with fixed points h_1 and h_2 such that*

- *satisfy a shared Lasota–Yorke inequality, i.e.,*

$$\left\|L_i^n f\right\|_s \leq A\lambda^n \|f\|_s + B\|f\|_w$$

for $i \in \{1, 2\}$
- *preserve the integral, i.e., $\int L_i f dx = \int f dx$ for $i = 1, 2$,*

and suppose moreover there is a C^* such that $\left\|L_i^n f\right\|_s \leq C^* \|f\|_s$, and for any $N \geq 1$ we have

$$\|h_1 - h_2\|_s \leq \|L_1^N(h_1 - h_2)\|_s + NC^*\|(L_1 - L_2)h_2\|_s.$$

Furthermore if $\left\| L_1^N|_{U^0} \right\|_s \leq C_N < 1$ *then we can have*

$$\|h_1 - h_2\|_s \leq \frac{NC^*\|(L_1 - L_2)h_2\|_s}{1 - C_N}.$$

Proof The value of C^* is given by $A\lambda + B$, and the distance between the two fixed points is shown as follows,

$$\begin{aligned}\|h_1 - h_2\|_s &\leq \|L_1^N h_1 - L_2^N h_2\|_s \\ &\leq \|L_1^N(h_1 - h_2)\|_s + \|(L_1^N - L_2^N)h_2\|_s.\end{aligned}$$

Note that

$$\begin{aligned}(L_1^N - L_2^N)h_2 &= \sum_{k=1}^{N} L_1^{N-k}(L_1 - L_2)L_2^{k-1}h_2 \\ &= \sum_{k=1}^{N} L_1^{N-k}(L_1 - L_2)h_2.\end{aligned}$$

Consequently,

$$\begin{aligned}\|(L_1^N - L_2^N)h_2\|_s &\leq \sum_{k=1}^{N} C^*\|(L_1 - L_2)h_2\|_s \\ &\leq NC^*\|(L_1 - L_2)h_2\|_s.\end{aligned}$$

Since L_1 and L_2 preserve the integral we have that $h_1 - h_2 \in U^0$ and therefore we can bound $\|L_1^N(h_1 - h_2)\|_s$ by $C_N\|h_1 - h_2\|_s$, rearranging gives us the last result. □

Remark 3.5 We remark that in order to use Lemma 3.4, our Chebyshev approximation scheme is adapted to preserve the integral, as explained in Remark 3.22.

Remark 3.6 The operators $\hat{L}$ and $\hat{L}_{\delta_k}$ admit a uniform Lasota–Yorke inequality,

$$\left\| \hat{L}^n f \right\|_{C^1} \leq A\lambda^n \|f\|_{C^1} + B\|f\|_{C^0}$$

as shown in Sect. 8.2.1, where a value for C^* is found. Bounds on $\left\| L_{\delta_k}|_{U^0} \right\|_{C^1}$ can be found by techniques described in Sect. 3.3.4. The C^1 norms of $\hat{h}_{\delta_k}$ and $\hat{h}$ can be estimated using the Lasota–Yorke inequalities. We can then use Lemma 3.3 to make the error in $\|\hat{h}_{\delta_k} - \hat{h}\|_{C^1}$ as small as we like.

Next we define a finite rank operator to obtain $\hat{h}_n$ so that $\|\hat{h}_n - \hat{h}_{\delta_k}\|_{C^1}$ can be made as small as required.

3.3 Approximating the Invariant Density for $\hat{T}_{\delta_k}$

To approximate the invariant density, we will discretize the operator $\hat{L}_{\delta_k}$ using the basis of the Chebyshev polynomials of the first kind; this approach is similar to [48], but instead of using a priori estimates on the spectral gap, we use a posteriori estimates on the mixing rate, in the spirit of [22, 26]. The Chebyshev basis is a basis for the space of polynomials with a main advantage: given a continuous function f on $[-1, 1]$ the interpolating polynomial on the Chebyshev points are "near-best" approximants with respect to $\|.\|_\infty$ [46, Theorem

16.1]; moreover if the function f is regular enough the coefficients of the interpolant decay "fast" and are easily computed by means of the Fast Fourier Transform.

Before going forward, some observations are in order, since Chebyshev polynomials do not solve all the problems involved with approximation: to apply this approximation scheme we need to prove A priori that our stationary density is regular enough and keep track of all the errors involved in the computation of the coefficients. Moreover, evaluating a Chebyshev polynomial of high degree rigorously is a delicate matter [37].

3.3.1 Chebyshev Interpolation and Projection

Given an $f \in W^{1,1}$ from $[-1, 1] \to \mathbb{C}$, we can define a function $\mathcal{F}(\theta)$ on $[0, 2\pi]$ by

$$\mathcal{F}(\theta) = f(\cos(\theta)).$$

By the Sobolev embedding theorem, we know that one of the representatives of the equivalence class f is absolutely continuous: in the following we assume this is the case.

Then, if we denote by $b_0 = a_0/2$, $b_{N-1} = a_{N-1}/2$ and $b_i = a_i$ for all $i = 1, \dots, N-2$:

$$p(x) = \sum_{k=0}^{N-1} b_k T_k(x),$$

where the a_k are the ones computed by the DFT is the interpolating polynomial of f on the grid given by the x_i.

Definition 3.7 We define the Fourier grid of $[0, 2\pi]$ of size $2N$ as $\theta_i = (2\pi i)/(2N)$ for $i = 0, \dots, 2N-1$.

The Discrete Fourier Transform (FFT) on a grid of size $2N$ allows us to compute the coefficients a_k of the trigonometric polynomial interpolating $\mathcal{F}$ on an equispaced grid $\theta_i = (2\pi i)/(2N)$, for i in $0, \dots, 2N-1$, i.e., for all i:

$$\mathcal{F}(\theta_i) = \sum_{j=-N+1}^{N} a_j e^{-i\pi j\theta_i}.$$

Remark that computing the coefficients a_k efficiently can be achieved by using the family of algorithms known as Fast Fourier Transforms, of which the most common is the Cooley-Tukey algorithm, which was first used by Gauss [30].

Definition 3.8 We define the Chebyshev grid of order N as $x_i = \cos(\theta_i)$ for i in $0, \dots, N$.

Denote by $\pi_x : \mathbb{C} \to R$ be the projection on the x axis; then, $\pi_x(e^{i\cdot\theta_i}) = x_i$. Observe that $\mathcal{F}(\theta_i) = \mathcal{F}(\theta_{2N-i}) = f(x_i)$.

Definition 3.9 We define the Chebyshev polynomials by the relation

$$T_n(\cos(\theta)) = \cos(n\theta).$$

Definition 3.10 Let $f \in W^{k,1}$, $k > 1$, we define the **Chebyshev (interpolating) projection**

$$\pi_n f = \sum_{k=0}^{n} b_k T_k(x).$$

where $b_0 = a_0/2$, $b_{N-1} = a_{N-1}/2$ and $b_i = a_i$ for all $i = 1, \ldots, N-2$ and the a_i are the coefficients computed by the DFT of $\mathcal{F}$.

Remark 3.11 If, instead of the FFT we had taken the Fourier transform of $\mathcal{F}$, the Fourier coefficients $\hat{a}_k$ would define coefficients $\hat{b}_k$, the Chebyshev orthogonal expansion

$$f(x) = \sum_{k=0}^{+\infty} \hat{b}_k T_k(x),$$

and the **Chebyshev projection**

$$\hat{\pi}_n f = \sum_{k=0}^{n} \hat{b}_k T_k(x).$$

The coefficients a_k and $\hat{a}_k$ are related by the aliasing relation:

$$a_k = \sum_{p\in\mathbb{Z}} \hat{a}_{k+p\cdot 2n},$$

a-priori knowledge of the regularity of f allows to estimate the aliasing error above.

This foundational Theorem from [46] estimates the decay rate of the Chebyshev coefficients.

Theorem 3.12 *For an integer $\nu \geq 0$, let f and its derivatives through $f^{(\nu-1)}$ be absolutely continuous on $[-1, 1]$ and suppose the ν^{th} derivative $f^{(\nu)}$ is of bounded variation V. Then for $k \geq \nu + 1$, the Chebyshev coefficients of f satisfy*

$$|\hat{b}_k| \leq \frac{2V}{\pi k(k-1)\ldots(k-\nu)} \leq \frac{2V}{\pi(k-\nu)^{\nu+1}}.$$

The decay rate of Chebyshev coefficients allows us to estimate the projection error in C^0 and C^1 norm.

Theorem 3.13 *If f satisfies the conditions of Theorem 3.12, with V again the total variation of $f^{(\nu)}$ for some $\nu \geq 1$, then for any $n \geq \nu$, its Chebyshev projection satisfies*

$$\|f - \pi_n f\|_\infty \leq \frac{2V}{\pi\nu n(n-1)\ldots(n+1-\nu)}$$

The following theorem is a consequence of combining theorem 3.12, with the proof of Theorem 2.3 from [51],

Theorem 3.14 *If $f, f', \ldots, f^{(\nu-1)}$ are absolutely continuous on $[-1, 1]$ and if $\left\|f^{(\nu)}\right\|_1 = V < \infty$ for some $\nu \geq 0$, then for each $n \geq \nu + 1$, we have that for $\nu > 2$*

$$\left\|f' - (\pi_n f)'\right\|_\infty \leq \frac{4(n+1)V}{n(\nu-2)\pi(n-2)(n-3)\ldots(n+1-\nu)}.$$

Proof From the proof of Theorem 2.3 from [51] we have

$$\left\|f' - (\pi_n f)'\right\|_\infty \leq 2\sum_{k=n+1}^{\infty} |a_k| k^2$$

and Theorem 3.12 then gives

$$\|f' - (\pi_n f)'\|_\infty \leq \sum_{k=n+1}^{\infty} \frac{4Vk^2}{\pi k(k-1)\dots(k-\nu)}$$
$$\leq \frac{4(n+1)V}{n(\nu-2)\pi(n-2)(n-3)\dots(n+1-\nu)}.$$

□

We can use these theorems to bound the error of Chebyshev projections in the C_1 norm.

3.3.2 Numerical Remarks: FFT and Chebyshev

It is important to have an explicit estimate of the error on the coefficients introduced by the FFT. The main issue here is that, computing Chebyshev points and evaluating the function f are not exact operation. To compute rigorous inclusions of the true mathematical value, we use Interval Arithmetics [47].

This means that we need to compute the FFT of a vector of intervals, not of floating point numbers. The following is the consequence of a classical result from [31] that allows us to find a vector of intervals that encloses the Fast Fourier Transform of any element of the vector of the values. This allows us to use optimized implementations of the FFT algorithm as FFTW [20].

Remark 3.15 We remark that, to use the following estimate as it is, we need to use Chebyshev basis such that the Chebyshev projection is computed by using a FFT with size a power of 2, i.e., the size of the Chebyshev basis is $2^{l-1}+1$ for $l > 1$, i.e., the Chebyshev polynomials have degree smaller than 2^{l-1}.

Remark 3.16 The estimate in [31] cited below can be generalized to other sizes; we stress that the precision of the Cooley–Tukey algorithm stems from the same origin as its computational effectiveness: its recursive structure that makes the operation count grow as $O(N\log(N))$.

Lemma 3.17 *Let $\tilde{f}$ be a vector of intervals of dimension $N = 2^l$, f_m the vector of their midpoints, f_r the vector of their radiuses. Let $\hat{a}$ be the computed FFT of f_m by the Cooley-Tukey FFT algorithm. Then*

$$\|\hat{a} - \bar{a}\|_2 \leq \frac{l}{\sqrt{N}}\left(\frac{\eta}{1-\eta}\|f_m\|_2 + \|f_r\|_2\right).$$

where $\bar{a}$ is the exact FFT for any $f \in \tilde{f}$, $\eta = \mu + \gamma_4(\sqrt{2}+\mu)$ with μ the absolute error in the computation of the twiddle factors and $\gamma_4 = 4u/(1-4u)$ where u is the unit roundoff.

Remark 3.18 It is important to stress that Floating Point numbers are not a field in the mathematical sense: due to rounding, floating point operations are neither associative nor commutative. This also implies that while, when acting on complex numbers, the Cooley-Tukey algorithm is a linear operator, on floating point numbers this is not true anymore. If $\hat{v}$ is the computed FFT of a vector v we must assume that in general $\widehat{(v+w)}$ is different from $\hat{v}+\hat{w}$; the estimate above is used to control the error arising from using the Cooley-Tukey algorithm in floating point arithmetic.

3.3.3 Bounding the Error on the Invariant Density

Let π_n be the Chebyshev projection and let

$$\hat{L}_n = \pi_n \hat{L}_{\delta_k} \pi_n$$

be our finite rank approximation of $\hat{L}_{\delta_k}$.

Lemma 3.19 *If $1/\hat{T}_{\delta_k}$ is of class C^ν then $\hat{L}_{\delta_k}$ admits Lasota–Yorke like inequalities of the form*

$$\left\|(\hat{L}^n_{\delta_k} f)^{(k)}\right\|_1 \leq (\lambda^k)^n \left\|f^{(k)}\right\|_1 + \tilde{A}_k \|f\|_{W^{k-1,1}},$$

for some $\nu \in \mathbb{N}$ for $k = 1, \ldots, \nu$. This implies that if h_{δ_k} is a fixed point of $\hat{L}_{\delta_k}$

$$\|h_{\delta_k}\|_{W^{k,1}} \leq \frac{A_k}{1-\lambda^k} \|h_{\delta_k}\|_{W^{k-1,1}}$$

Remark 3.20 The Lasota–Yorke inequalities give us an upper bound on the $W^{k,1}$ norm of the fixed point. This, together with Theorems 3.13 and 3.14 permit us to control the discretization error. To estimate our error, we need to compute the constants of this Lasota–Yorke inequality explicitly, we refer to Sect. 7.2.3 for the technique we used.

Remark 3.21 We can use theorems 3.13 and 3.14, together with Lemma 3.19 to get a bound on $\|\hat{L}_{\delta_k} - \hat{L}_n\|_{C^1}$ and the techniques from Sect. 3.3.4 to with Lemma 3.4 using operators $\hat{L}_{\delta_k}$ and $\hat{L}_n$ in order to bound $\left\|\hat{h}_{\delta_k} - \hat{h}_n\right\|_{C^1}$ explicitly. This approach is now quite established, a full treatment can be found in [22, 26].

Remark 3.22 The discretized operator obtained by the Chebyshev discretization does not preserve the value of the integral. To solve this issue, as in [26] we correct the behaviour of the discretized operator by defining a new operator

$$\hat{Q}_\eta f = \hat{L}_\eta f + 1 \cdot (\int f dx - \int \hat{L}_\eta f dx)$$

which is guaranteed to preserve the space of average 0 measure and has eigenvalue 1, since the row vector that contains the integrals of the basis elements is preserved by multiplication on the left.

3.3.4 Numerical Remarks: Convergence Rates

The problem of bounding the error in C^1 is now reduced to estimate c_N such that $\left\|\hat{L}^N_n|_{U^0}\right\|_{C^1} \leq c_N$. Since the operator L_n is of finite rank, we can use numerical methods to compute these quantities in a rigorous way.

Given a basis $\{e_i\}_i$ of U^0 normalized with respect to the C^1 norm, a generic function v in U^0 is written as $v = \sum_{i=1}^N b_i e_i$. We want to find a constants C_k such that

$$\|\hat{L}^k_n v\|_{C^1} \leq C_k \|v\|_{C^1}.$$

If a c_k exists such that for all basis elements e_i we have

$$\|\hat{L}^k_n e_i\|_{C^1} \leq c_k.$$

then

$$\|\hat{L}_n^k v\|_{C^1} \leq c_k \sum_{i=1}^{N} |b_i| \leq c_k \|b\|_{\ell^1}$$

where the ℓ_1 norm is the linear algebra norm on the coefficients; we will exhibit a constant D such that $\|.\|_{\ell^1} \leq D\|.\|_{C^1}$; then $C_k \leq Dc_k$.

A basis of U^0 in the Chebyshev basis is given by

$$e_i = \frac{g_i}{\|g_i\|_{C^1}}, \quad \text{where} \quad g_i(x) = T_i(x) - \int_{-1}^{1} T_i(x)dx.$$

We can link the Chebyshev coefficients a_i and b_i by

$$a_0 = -\sum_{i=1}^{N} b_i \int_{-1}^{1} T_i(x)dx$$

and $a_i = b_i$ for $i > 0$. We can use Theorem 3.12 to say

$$|a_k| \leq \frac{\|v\|_{C^1}}{k}$$

and therefore

$$\|b\|_{\ell^1} = \sum_{k=1}^{N} |b_i| \leq \sum_{k=1}^{N} \frac{\|v\|_{C^1}}{k} \leq \log(N+1)\, \|v\|_{C^1}$$

so we have $D = \log(N+1)$ and

$$\left\| \hat{L}_n^k v \right\|_{C^1} \leq c_k \log(N+1)\, \|v\|_{C^1}.$$

Computationally if we take N functions

$$\hat{e}_i = \frac{g_i}{\|g_i\|_{C^1}^-},$$

where $\|g_i\|_{C^1}^-$ is a lower bound on the C^1 norm of g_i and if we calculate each

$$\left\| \hat{L}_n^k e_i \right\|_{C^1}$$

then take the maximum value and call it $\hat{c}_k$, then $\hat{c}_k \log(N+1)$ is an upper bound on the C^1 contraction of $\left\| \hat{L}_n^k|_{U^0} \right\|_{C^1}$.

It is important to explain how we compute an upper bound for the C^1 norm: we use a classical optimization algorithm in IntervalArithmetics [47] that allows us to give a certified upper bound, implemented in the Julia package IntervalOptimisation.jl. The main issue here is that the Clenshaw algorithm is prone to overestimation when evaluated on intervals [37]; to solve this we extended the algorithms in [37] to get tighter bound for the maximum of a Chebyshev polynomial and its derivative.

4 Approximating the Linear Response for the Induced System

We now provide an approximation of the linear response of the induced map, $\hat{h}^*$, in the L^1-norm, through the use of the Ulam approximation; we refer to [22, 26] for an in-depth treatment of the Ulam discretization. We recall that the functions A_0 and $B = 0$ are defined as

$$A_0 = -\left(\frac{\partial_\epsilon \hat{T}_\epsilon}{\hat{T}'_\epsilon}\right)\Big|_{\epsilon=0}, \qquad B_0 = \left(\frac{\partial_\epsilon \hat{T}_\epsilon \cdot \hat{T}''_\epsilon}{\hat{T}'^2_\epsilon} - \frac{\partial_\epsilon \hat{T}'_\epsilon}{\hat{T}'_\epsilon}\right)\Big|_{\epsilon=0}.$$

To compute the linear response we need to approximate the quantity (2.10)

$$\hat{h}^* = (I - \hat{L})^{-1}\hat{L}[A_0\hat{h}' + B_0\hat{h}],$$

by using the Neumann inequality and the spectral gap of $\hat{L}$ we pretend to approximate this by a finite sum

$$\sum_{i=0}^{K} \hat{L}^i \hat{L}[A_0\hat{h}' + B_0\hat{h}],$$

but this involves some delicate work, the first step being approximating $\hat{L}$ by $\hat{L}_{\delta_k}$; while $\hat{L}[A_0\hat{h}' + B_0\hat{h}]$ is bounded, we have that $\hat{L}_{\delta_k}[A_0\hat{h}' + B_0\hat{h}]$ is unbounded, therefore is not well representable in the Chebyshev basis. Therefore, we need to resort to a different basis for the discretization of L_{δ_k}, the Ulam basis.

Definition 4.1 The Ulam projection, is a projection $\Pi_u : L^1([0.5, 1]) \to L^1([0.5, 1])$ over a partition of $[0.5, 1]$, denoted by $\mathcal{P}$,

$$\Pi_u f(x) = \frac{1}{|A|}\int_A f(x)dx$$

where $x \in A \in \mathcal{P}$. The Ulam discretisation of the transfer operator L_{δ_k} is

$$\hat{L}_u := \Pi_u \hat{L}_{\delta_k} \Pi_u.$$

4.1 Error in Approximating the Linear Response

Set our approximation of (2.10) to be

$$\hat{h}^*_u := \sum_{n=0}^{l^*} \hat{L}^n_u \hat{L}_{\delta_k}[A_0\hat{h}'_n + B_0\hat{h}_n]. \tag{4.1}$$

where $\hat{L}_u$ is the Ulam approximation of $\hat{L}_{\delta_k}$.

Lemma 4.2 *We have*

$$\begin{aligned}\|\hat{h}^*_u - \hat{h}^*\|_1 \le &\sum_{n=0}^{l^*}\sum_{i=0}^{n} \left\|(\hat{L}_u - \hat{L}_{\delta_k})\hat{L}^{n-i}_u W_u\right\|_1 + \sum_{n=0}^{l^*}\sum_{i=0}^{n} \left\|(\hat{L}_{\delta_k} - \hat{L})\hat{L}^{n-i}_u W_u\right\|_1 \\ &+ l^* \|(W_u - W)\|_1 + \sum_{n=l^*+1}^{\infty} \left\|\hat{L}^n W\right\|_1\end{aligned} \tag{4.2}$$

where $W = \hat{L}[A_0\hat{h}' + B_0\hat{h}]$ *and*

$$W_u(x) = \sum_{\omega\le k} \frac{1}{\eta}\int_{g_\omega(I_i)} A_0(z)\cdot \hat{h}_n'(\zeta) + B_0(z)\hat{h}_n(\zeta)dz + \frac{1}{\eta}\int_{\delta_k(2I_i-1)+1/2}[A_0\hat{h}_n' + B_0\hat{h}_n](z)dz$$

for $x \in I_i$ *and* $\zeta \in g_\omega(I_i)$.

Proof First we must recall from (4.1) that

$$\hat{h}_u^* = \sum_{n=0}^{l^*} \hat{L}_u^n \Pi_u L_{\delta_k}[A_0\hat{h}_n' + B_0\hat{h}_n].$$

Take $x \in I_i$, then we have

$$\begin{aligned}
&\Pi_u L_{\delta_k}[A_0\hat{h}_n' + B_0\hat{h}_n](x)\\
&= \frac{1}{\eta}\int_{I_i}\sum_{\omega\le k}[A_0\hat{h}_n' + B_0\hat{h}_n]\circ g_\omega(y)g_\omega'(y) + [A_0\hat{h}_n' + B_0\hat{h}_n]\circ(\delta_k(2y-1)+1/2)\delta_k dy\\
&= \frac{1}{\eta}\sum_{\omega\le k}\int_{g_\omega(I_i)}[A_0\hat{h}_n' + B_0\hat{h}_n](z)dz + \frac{1}{\eta}\int_{\delta_k(2I_i-1)+1/2}[A_0\hat{h}_n' + B_0\hat{h}_n](z)dz.
\end{aligned}$$

The estimate follows by direct calculation. Indeed, by (2.10) and the definition of $\hat{h}_u^*$ we have

$$\begin{aligned}
\left\|\hat{h}^* - \hat{h}_u^*\right\|_1 &= \left\|\sum_{n=0}^{\infty}\hat{L}^n W - \sum_{n=0}^{l^*}\hat{L}_u^n W_u\right\|_1\\
&\le \left\|\sum_{n=0}^{l^*}\hat{L}^n W - \sum_{n=0}^{l^*}\hat{L}_u^n W_u\right\|_1 + \sum_{n=l^*+1}^{\infty}\left\|\hat{L}^n W\right\|_1
\end{aligned}$$

which, by using the triangle inequality and the second resolvent identity, is bounded above by

$$\begin{aligned}
&\left\|\sum_{n=0}^{l^*}\hat{L}^n W_u - \sum_{n=0}^{l^*}\hat{L}_u^n W_u\right\|_1 + \left\|\sum_{n=0}^{l^*}(\hat{L}^n W - \hat{L}^n W_u)\right\|_1 + \sum_{n=l^*+1}^{\infty}\left\|\hat{L}^n W\right\|_1\\
&\le \sum_{n=0}^{l^*}\sum_{i=0}^{n}\left\|\hat{L}^i(\hat{L}_u - \hat{L})\hat{L}_u^{n-i}W_u\right\|_1 + \sum_{n=0}^{l^*}\left\|\hat{L}^n(W_u - W)\right\|_1 + \sum_{n=l^*+1}^{\infty}\left\|\hat{L}^n W\right\|_1\\
&\le \sum_{n=0}^{l^*}\sum_{i=0}^{n}\left\|(\hat{L}_u - \hat{L}_{\delta_k})\hat{L}_u^{n-i}W_u\right\|_1 + \sum_{n=0}^{l^*}\sum_{i=0}^{n}\left\|(\hat{L}_{\delta_k} - \hat{L})\hat{L}_u^{n-i}W_u\right\|_1\\
&\quad + l^*\left\|(W_u - W)\right\|_1 + \sum_{n=l^*+1}^{\infty}\left\|\hat{L}^n W\right\|_{C^1}
\end{aligned}$$

□

Remark 4.3 The estimates in Lemma 4.2 can all be made as small as desired. Indeed, notice that $\hat{L}[A_0\hat{h}' + B_0\hat{h}]$ is a zero average C^1 function; therefore

- the last summand $\sum_{n=l^*+1}^{\infty} \|\hat{L}^{n+1} W\|_{C^1}$ can be made, for sufficiently large l^*, small since $\hat{L}$ admits a spectral gap when acting on C^1. Once this term is estimated, l^* is fixed once and for all;
- the summand $\sum_{n=0}^{l^*} \sum_{i=1}^{n} \|(\hat{L} - \hat{L}_{\delta_k})\hat{L}_u^{n-i} W_u\|_1$ can be made small by choosing δ_k small enough;
- the summand $\sum_{n=0}^{l^*} \sum_{i=1}^{n} \|(\hat{L}_{\delta_k} - \hat{L}_u)\hat{L}_u^{n-i} W_u\|_1$ can be made small by choosing η, the size of the Ulam discretization, small enough;
- the term $l^* \|(W_u - W)\|_1$ can be made small by reducing δ_k.

5 Normalising the Density and the Linear Response

Ultimately the goal is to approximate the dynamics of the system, so we would like the invariant measure to be a probability measure. This is not always possible for maps with indeterminate fixed points, however it was shown in [11] that a fixed point of the transfer operator of an LSV map is bounded above by $Cx^{-\alpha}$, for some constant C, giving a maximum integral of $\frac{C}{1-\alpha}$, so we can make our calculated density a probability density by normalising with respect to its integral. Our approximation of the density, h_η, when normalised will then have an error of

$$\left\| \frac{h_\eta}{\int h_\eta dm} - \frac{h}{\int h dm} \right\|_1 = \left\| \frac{h_\eta}{\int h_\eta dm} - \frac{h}{\int h_\eta dm} + h\frac{\int h dm - \int h_\eta dm}{\int h dm \int h_\eta dm} \right\|_1$$
$$\leq \frac{\|h - h_\eta\|_1}{\int h_\eta dm} + \frac{\|h\|_1}{\int h dm}\frac{\|h - h_\eta\|_1}{\int h_\eta dm} \leq 2\frac{\|h - h_\eta\|_1}{\int h_\eta dm}$$

and if we ensure that the integral is preserved throughout the approximation then the error is $\frac{\|h-h_\eta\|_1}{\int h_\eta dm}$.

Since the Chebyshev approximation does not preserve the integral we use the first estimate to bound

$$\left\| \frac{h_\eta}{\int h_\eta dm} - \frac{h}{\int h dm} \right\|_1$$

where we calculate the integral of h_η by

$$\int_0^1 F^{app} \hat{h}_\eta dx = \int_{0.5}^1 \hat{h}_\eta dx + \sum_{|\omega|=1}^{N^*} H_\eta \circ g_\omega(1) - H_\eta \circ g_\omega(0.5)$$

where $H_\eta(x) = \int_{0.5}^x h_\eta dx$.

The linear response for the normalised invariant density is then ρ^* such that

$$\lim_{\epsilon \to 0} \left\| \frac{\frac{h}{\int h dm} - \frac{h_\epsilon}{\int h_\epsilon dm}}{\epsilon} - \rho^* \right\|_1 = 0$$

we get from this

$$\left\| \frac{\frac{h}{\int hdm} - \frac{h_\epsilon}{\int h_\epsilon dm}}{\epsilon} - \rho^* \right\|_1 = \left\| \frac{\frac{h}{\int hdm} - \frac{h_\epsilon}{\int h+\epsilon h^* + o(\epsilon^2) dm}}{\epsilon} - \rho^* \right\|_1$$
$$= \left\| \frac{\frac{h}{\int hdm} - \frac{h_\epsilon}{\int hdm}}{\epsilon} - \frac{h_\epsilon \int [h^* + o(\epsilon)]dm}{\int hdm (\int [h + \epsilon h^* + o(\epsilon^2)]dm)} - \rho^* \right\|_1$$

which tells us that

$$\rho^* = \frac{h^*}{\int hdm} - \frac{h \int h^* dm}{(\int hdm)^2}.$$

For brevity later let

$$A = \frac{h^*}{\int hdm} - \frac{h^*_\eta}{\int h_\eta dm} \quad B = -\frac{h \int h^* dm}{(\int hdm)^2} + \frac{h_\eta \int h^*_\eta dm}{(\int h_\eta dm)^2}.$$

The error on the normalised linear response is calculated as follows,

$$\left\| \frac{h^*}{\int hdm} - \frac{h \int h^* dm}{(\int hdm)^2} - \frac{h^*_\eta}{\int h_\eta dm} + \frac{h_\eta \int h^*_\eta dm}{(\int h_\eta dm)^2} \right\|_1$$

focusing on the first part for now,

$$= \left\| \frac{h^*}{\int h_\eta dm} - \frac{h^*_\eta}{\int h_\eta dm} + \frac{h^* \int (h_\eta - h)dm}{\int hdm \int h_\eta dm} + B \right\|_1$$
$$= \left\| \frac{h^* - h^*_\eta}{\int h_\eta dm} + \frac{h^* \int (h_\eta - h)dm}{\int hdm \int h_\eta dm} + B \right\|_1$$

and now on the second part

$$= \left\| A + \frac{h_\eta \int h^*_\eta dm}{(\int h_\eta dm)^2} - \frac{h \int h^* dm}{(\int hdm)^2} \right\|_1$$
$$\leq \left\| A + \frac{h_\eta \int h^*_\eta dm}{(\int h_\eta dm)^2} - \frac{(h_\eta + (h - h_\eta))(\int h^*_\eta dm - \left\| h^* - h^*_\eta \right\|_1)}{(\int h_\eta dm + \left\| h - h_\eta \right\|_1)^2} \right\|_1$$
$$= \left\| A - \frac{(h - h_\eta)(\int h^*_\eta dm - \left\| h^* - h^*_\eta \right\|_1) - h_n \left\| h^* - h^*_\eta \right\|_1}{(\int h_\eta dm + \left\| h - h_\eta \right\|_1)^2} \right.$$
$$\left. + \frac{h_\eta \int h^*_\eta dm (2 \int h_\eta dm \left\| h - h_\eta \right\|_1 + \left\| h - h_\eta \right\|_1^2)}{(\int h_\eta dm + \left\| h - h_\eta \right\|_1)^2 (\int h_\eta)^2} \right\|_1.$$

This allows us to bound the L^1 error of the normalised linear response by

$$\begin{aligned}
&\frac{\left\|h^*-h^*_\eta\right\|_1}{\left\|h_\eta\right\|_1}+\frac{(\left\|h^*_\eta\right\|_1+\left\|h^*-h^*_\eta\right\|_1)\left\|h-h_\eta\right\|_1}{(\left\|h_\eta\right\|_1-\left\|h-h_\eta\right\|_1)\left\|h_\eta\right\|_1}\\
&+\frac{\left\|h-h_\eta\right\|_1(\left\|h^*_\eta\right\|_1-\left\|h^*-h^*_\eta\right\|_1)}{(\left\|h_\eta\right\|_1+\left\|h-h_\eta\right\|_1)^2}+\frac{\left\|h_\eta\right\|_1\left\|h^*-h^*_\eta\right\|_1}{(\left\|h_\eta\right\|_1+\left\|h-h_\eta\right\|_1)^2}\\
&+\frac{2\left\|h^*_\eta\right\|_1\left\|h-h_\eta\right\|_1}{(\left\|h_\eta\right\|_1+\left\|h-h_\eta\right\|_1)^2}+\frac{\left\|h^*_\eta\right\|_1\left\|h-h_\eta\right\|_1^2}{(\left\|h_\eta\right\|_1+\left\|h-h_\eta\right\|_1)^2\left\|h_\eta\right\|_1}. \qquad (5.1)
\end{aligned}$$

6 Proof of Theorem 2.18

In this section we give a proof of the main result in the paper, i.e., that we can approximate as well as we want the linear response.

Proof of Theorem 2.18 Using (2.12) we have

$$\begin{aligned}
\|h^*-h^*_\eta\|_1 &\le \|F_0\hat{h}^*-F_0^{\text{app}}\hat{h}^*_u\|_1+\|Q\hat{h}-Q^{\text{app}}\hat{h}_n\|_1\\
&\le \|F_0\hat{h}^*-F_0\hat{h}^*_u\|_1+\|F_0\hat{h}^*_u-F_0^{\text{app}}\hat{h}^*_u\|_1\\
&\quad+\|Q\hat{h}-Q\hat{h}_n\|_1+\|Q\hat{h}_n-Q^{\text{app}}\hat{h}_n\|_1 \quad := (I+II+III+IV).
\end{aligned} \qquad (6.1)$$

By (2.6), we get

$$\begin{aligned}
(I) &\le \|\hat{h}^*-\hat{h}^*_u\|_1+\sum_{\omega\in\Omega}\int_{\Delta^c}\left|\left(\hat{h}^*\circ g_\omega-\hat{h}^*_u\circ g_\omega\right)g'_\omega\right|dx\\
&\le \|\hat{h}^*-\hat{h}^*_u\|_1+\|\hat{h}^*-\hat{h}^*_u\|_{C^0}\cdot\sum_\omega\|g'_\omega\|_{\mathcal{B}}\int_{\Delta^c}x^{-\gamma}dx.
\end{aligned} \qquad (6.2)$$

Using (2.6) again, we have

$$\begin{aligned}
\text{(II)} &\le \sum_{|[\omega]|>N^*}\int_{\Delta^c}\left|\hat{h}^*_u\circ g_\omega g'_\omega\right|dx \le \|\hat{h}^*_u\|_{C^0}\cdot\sum_{|[\omega]|>N^*}\|g'_\omega\|_{\mathcal{B}}\int_{\Delta^c}x^{-\gamma}dx\\
&\le \frac{1}{2^{1-\gamma}(1-\gamma)}\cdot\|\hat{h}^*_u\|_{C^0}\cdot\sum_{|[\omega]|>N^*}\|g'_\omega\|_{\mathcal{B}}.
\end{aligned} \qquad (6.3)$$

Note that by (2.4), one can choose N^* large enough so that (II) is sufficiently small. Using (2.9), we have

$$(III)\le\sum_{\omega\in\Omega}\int_{\Delta^c}\left|\left(\hat{h}'\circ g_\omega-\hat{h}'_n\circ g_\omega\right)\cdot a_\omega g'_\omega\right|dx+\sum_{\omega\in\Omega}\int_{\Delta^c}\left|\left(\hat{h}\circ g_\omega-\hat{h}_n\circ g_\omega\right)\cdot b_\omega\right|dx.$$

Now using (2.3), (2.5), and the change of variables $y_\omega = g_\omega(x)$ we get

$$
\begin{aligned}
(III) &\leq \sup_\omega |a_\omega| \sum_{\omega\in\Omega} \int_{[\omega]_0} \left|\hat{h}'(y_\omega) - \hat{h}'_n(y_\omega)\right| dy_\omega + \|\hat{h} - \hat{h}_n\|_{C^0} \cdot \sum_{\omega\in\Omega} \|b_\omega\|_{\mathcal{B}} \int_{\Delta^c} x^{-\gamma} dx \\
&= \sup_\omega |a_\omega| \cdot \|\hat{h}' - \hat{h}'_n\|_1 + \frac{1}{2^{1-\gamma}(1-\gamma)} \cdot \sum_{\omega\in\Omega} \|b_\omega\|_{\mathcal{B}} \cdot \|\hat{h} - \hat{h}_n\|_{C^0} \qquad (6.4) \\
&\leq \max\left\{\sup_\omega |a_\omega|, \frac{1}{2^{1-\gamma}(1-\gamma)} \cdot \sum_{\omega\in\Omega} \|b_\omega\|_{\mathcal{B}}\right\} \cdot \|\hat{h} - \hat{h}_n\|_{C^1}.
\end{aligned}
$$

Finally, using (2.9) again, we have

$$
(IV) \leq \sum_{|[\omega]|>N^*} \int_{\Delta^c} \left|\hat{h}'_n \circ g_\omega \cdot a_\omega g'_\omega\right| dx + \sum_{|[\omega]|>N^*} \int_{\Delta^c} \left|\hat{h}_n \circ g_\omega \cdot b_\omega\right| dx.
$$

Using (2.3) and (2.4) in the first integral, and using (2.5) in the second integral, we choose N^* large enough and get

$$
(IV) \leq \frac{1}{2^{1-\gamma}(1-\gamma)} \left[\sup_\omega |a_\omega| \cdot \|\hat{h}'_n\|_{C^0} \cdot \sum_{|[\omega]|>N^*} \|g'_\omega\|_{\mathcal{B}} + \|\hat{h}_n\|_{C^0} \cdot \sum_{|[\omega]|>\in N^*} \|b_\omega\|_{\mathcal{B}}\right]. \qquad (6.5)
$$

Choosing l^* in 4.1 to make $\sum_{n=l^*+1} \left\|\hat{L}^n W\right\|_{C^1}$ small enough, followed by k and η to make (6.2) and (6.4) small enough, then choosing N^* in (6.3) and (6.5) so $(I+II+III+IV) \leq \tau$ completing the proof. □

7 Application to an Example

In this section we will apply our algorithm to a classical example of maps with an indifferent fixed point, strictly related to Pomeau–Manneville maps, the Liverani–Saussol–Vaienti map. The behaviour of this map is determined by the exponent α; if $\alpha \in (0, 1)$ it is a non-uniformly expanding map with an absolutely continuous invariant probability measure; if $\alpha \geq 1$ there is an absolutely continuous invariant infinite measure.

7.1 Definition of the Map and the Induced Map

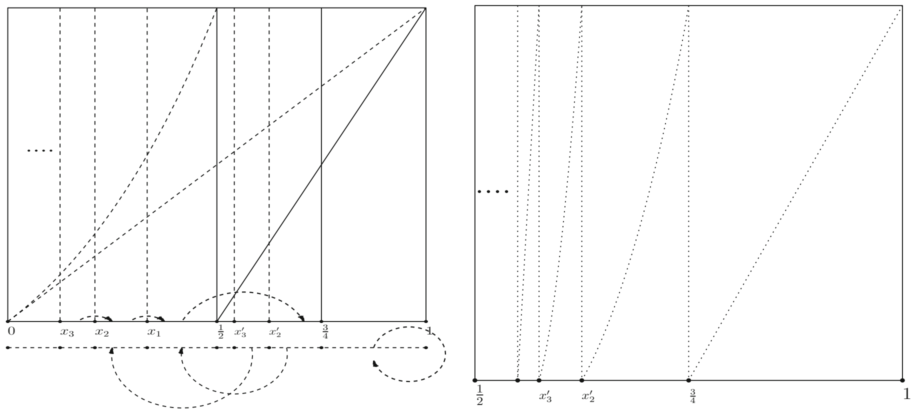
The equation of the map is

$$
T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) \text{ if } x \in [0, \frac{1}{2}] \\ 2x - 1 \text{ if } x \in (\frac{1}{2}, 1] \end{cases}. \qquad (7.1)
$$

Numerical assumption 7.1 We fix $\alpha = 0.125$ in our example. This is the value corresponding to $\epsilon = 0$ in the previous section.

We construct the inducing scheme as in Sect. 2.1. Let $x'_0 = 1$, $x'_1 = \frac{3}{4}$, and

$$
x'_n = g_\omega\left(\frac{1}{2}\right) \quad \text{for } n \geq 2.
$$



(A) The map T (B) The induced map $\hat{T}$

Fig. 2 The inducing scheme for T in (7.1)

Letting $\omega = 10^n$ and $g_\omega = g_1 \circ g_0^n$, then cylinder set $[\omega]$ is given by $g_\omega([0.5, 1]) = (x'_n, x'_{n-1}]$.

Then $\hat{T} : \Delta \to \Delta$ is a piecewise smooth and onto map with countable number of branches and it satisfies all the assumptions of Sect. 2.1. See Fig. 2 for a pictorial representation of the above inducing scheme.

7.1.1 Numerical Remark: The Shooting Method

To approximate rigorously the operators in this paper we need a rigorous way to approximate long orbits given a coding, i.e., we need to be able to compute

$$x = g_\omega(y) = g_1 \circ g_0^{n-1}(y)$$

i.e. we need to be able to compute $x \in [0.5, 1]$ such that

$$T^{n+1}(x) = y, \quad T^i(x) \in [0, 0.5]$$

for $i \in \{1, \ldots, n\}$.

To solve this problem efficiently and obtain tight bounds on x is tricky because taking preimages sequentially leads to propagation of errors and the computed interval ends up being not usable.

The main idea is to substitute the equation above with the following system of equations (this technique is called the Shooting Method, and we were introduced to it by W. Tucker)

$$\begin{cases} T(x_1) - x_2 = 0 & x_1 \in [0.5, 1] \\ T(x_2) - x_3 = 0 & x_2 \in [0, 0.5] \\ T(x_3) - x_4 = 0 & x_3 \in [0, 0.5] \\ \quad\vdots & \\ T(x_n) - y = 0 & x_n \in [0, 0.5]. \end{cases}$$

We will use the rigorous Newton method [47], to simultaneously enclose the points $x_1, \ldots, x_n$ that satisfy the system of equations above.

This method is one of the working horses of Validated Numerics. Given a function f with nonzero derivative on an interval I, the Interval Newton method allows us to prove, with the

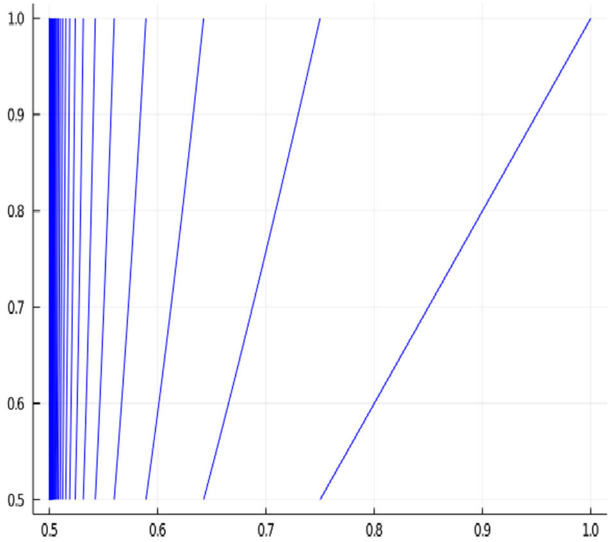


Fig. 3 The approximation map with $k = 200$

assistance of a computer, if the function f has a 0 in I and find an interval $J \subset I$ which is representable on the computer and contains the zero of the function. This generalizes to multivariate functions, using the formula in equation (7.2).

This way we are solving a unique system of equations instead of propagating backwards the error through solving equations with a "fat" variable. Given a function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and a vector of intervals $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the rigorous Newton step is given by

$$N(\hat{x}) = \hat{x} \cap (\mathrm{mid}(\hat{x}) - D\phi(\hat{x})^{-1}\phi(\mathrm{mid}(\hat{x}))), \tag{7.2}$$

where the intersection between interval vectors is meant componentwise and mid is a function that sends a vector of intervals to the vector of their midpoints [47].

In our specific case, the shooting method is numerically well behaved: denoting by $\phi(x_1, \ldots, x_n) = (T(x_1) - x_2, \ldots, T(x_n) - y)^T$, the Jacobian $D\phi$ is given by a bidiagonal matrix, whose $i - th$ diagonal entry is $T'(x_i)$ and the superdiagonal entries are constant and equal to -1. In particular, this guarantees us that the Jacobian is invertible, since its eigenvalues correspond to the diagonal elements and these are bounded away from 0. Moreover a bidiagonal system is solved in time $O(n)$ by backsubstitution, with small numerical error, and these assumptions guarantee that the interval Newton method converges.

This allows us to compute tight enclosure of $g_\omega(y)$, $g'_\omega(y)$, which allows us to compute discretizations of the transfer operator.

7.2 Computing the Error When Taking a Finite Number of Branches

Since we cannot calculate values for maps with infinitely many branches on the computer we use an approximating map as described in Sect. 3.2, this is depicted in Fig. 3 for $\alpha = 0.125$. To calculate bounds on the C^1 distance between the systems these maps define we use Lemma 3.3 and find D and D_0 for LSV maps. Estimating these bounds efficiently is delicate since it involves estimating the sum (and the tail) of converging series whose general term is going to zero slowly. The estimates in literature [4, 36] give rise to values that are impractical for our computations; as an example, the value of the constant C_8 in [36] computed according to their proof is of the order of 10^{269}, which makes its use in our computations unfeasible, therefore some work is needed to give sharper bounds for the constants. Since these estimates are quite technical and need the introduction of specific notations, we separate them in Appendix not to hinder the flow of the sections.

In Sect. 8.2 we bound $D_0 \leq 2.956$ and $D \leq 19.22$, so

$$\left\|(\hat{L} - \hat{L}_{\delta_k})\right\|_{C^1} \leq (D + D_0 D + 2)\delta_k \leq 3.609\delta_k$$

can be made as small as needed by increasing k.

Choosing $k = 200$ gives $\left\|\hat{L} - \hat{L}_{\delta_k}\right\|_{C^1} \leq 7.743 \cdot 10^{-12}$.

7.2.1 Bounding $\left\|\hat{h}_{\delta_k} - \hat{h}\right\|_{C^1}$

In Sect. 8.2.1 we prove the following Lasota–Yorke inequality

$$\left\|\hat{L}^n_{\delta_k} f\right\|_{C^1} \leq 2.491 \cdot (0.5)^n \|f\|_{C^1} + 6.206 \|f\|_\infty$$

.

The Lasota–Yorke inequality implies that

$$\left\|\hat{L}^n_{\delta_k} f\right\|_{C^1} \leq 7.452,$$

which together with Lemma 3.6, and the fact proved in Sect. 7.3.5 by using the methods from [25] that

$$\left\|\hat{L}^4_{\delta_k}|_{U^0}\right\|_{C^1} \leq 0.1557$$

allowing us to bound

$$\left\|\hat{h} - \hat{h}_{\delta_k}\right\|_{C^1} \leq \frac{4 \cdot 7.452}{1 - 0.1557} \left\|(\hat{L}_{\delta_k} - \hat{L})\hat{h}\right\|_{C^1} \leq 2.734 \cdot 10^{-10} \left\|\hat{h}\right\|_{C^1} \tag{7.3}$$

Observing that $\left\|\hat{h}\right\|_{C^1} \leq \left\|\hat{h}_{\delta_k}\right\|_{C^1} + \left\|\hat{h} - \hat{h}_{\delta_k}\right\|_{C^1}$ and a bound on $\left\|\hat{h}_{\delta_k}\right\|_{C^1}$ in Sect. 8.2.1 gives us a final error of $2.113 \cdot 10^{-9}$.

7.2.2 Computing the Discretization Error

The truncated operator $\hat{L}_{\delta_k}$ satisfies the following Lasota–Yorke like inequalities[5]

$$\|(\hat{L}^n_{\delta_k}) f'\|_1 \leq 0.5^n \|f'\|_1 + 1.785\|f\|_1$$
$$\|(\hat{L}^n_{\delta_k}) f''\|_1 \leq 0.5^n \|f''\|_1 + 0.3076\|f'\|_1 + 6.57\|f\|_1$$
$$\|(\hat{L}_{\delta_k} f)'''\|_1 \leq 0.5^{3n} \|f'''\|_1 + 0.145\|f''\|_1 + 1.98\|f'\|_1 + 36.96\|f\|_1$$
$$\|(\hat{L}^n_{\delta_k} f)^{(4)}\|_1 \leq 0.5^{4n} \|f^{(4)}\|_1 + 0.057\|f^{(3)}\|_1 + 1.49\|f''\|_1 + 16.97\|f'\|_1 + 559.4\|f\|_1$$
$$\|(\hat{L}^n_{\delta_k} f)^{(5)}\|_1 \leq 0.5^{5n} \|f^{(5)}\|_1 + 0.0199\|f^{(4)}\|_1 + 0.85\|f^{(3)}\|_1$$

[5] it is straightforward to see that these inequalities imply Lasota–Yorke inequalities on $W^{k,1}$ with weak norm $W^{k-1,1}$.

Table 1 Calculated contraction rates of our discretised operators

k	$\left\Vert \hat{L}_n^k\vert_{U^0}\right\Vert_{C^1}$	$\left\Vert \hat{L}_u^k\vert_{U^0}\right\Vert_1$
1	3.674	1
2	1.254	1
3	0.4237	1
4	0.1427	1
5	0.04799	1
6	0.01613	1
7	0.005421	1
8	0.001821	1
9	0.0006119	1
10	0.0002056	0.09782
11	0.0007551	0.02349

$$
\begin{aligned}
&+ 17.57\|f''\|_1 + 794.59\|f'\|_1 + 10086\|f\|_1\\
\|(\hat{L}^n_{\delta_k} f)^{(6)}\|_1 \leq& 0.5^{6n}\|f^{(6)}\|_1 + 0.0066\|f^{(5)}\|_1 + 0.41\|f^{(4)}\|_1\\
&+ 13.33\|f^{(3)}\|_1 + 895\|f''\|_1 + 24840.2\|f'\|_1 + 684431\|f\|_1.
\end{aligned}
$$

Since we know $\left\Vert h_{\delta_k}\right\Vert_1 = 1$ for f a probability density, we can use these to get a bound on $\left\Vert h^{(6)}_{\delta_k}\right\Vert_1$ which is calculated to be $7.953 \cdot 10^5$. Denoting by $\hat{L}_n = \pi_n \hat{L}_{\delta_k}\pi_n$ the discretized operator on the base of Chebyshev polynomials of the first kind of degree up to n, the same Lasota–Yorke inequalities allow us to compute

$$\left\Vert \hat{L}_{\delta_k} - \hat{L}_n\right\Vert_{C^1} \leq 3.297 \cdot 10^{-11}.$$

This, together with the computed bounds on the C^1 mixing rate in table 1 gives us an error of

$$\left\Vert \hat{h} - \hat{h}_n\right\Vert_{C^1} \leq 3.833 \cdot 10^{-9};$$

in figure 4 a plot of the approximated density is presented.

7.2.3 Numerical Remark: Automated Lasota–Yorke Inequalities

We detail a way to automatically calculate Lasota–Yorke type inequalities for transfer operators in $W^{k,1}$. Following [13] let

$$L_k f = \sum_{y\in T^{-1}(x)} \frac{f(y)}{|T'(y)|^k}.$$

From this follows

$$(L_k f)' = L_{k+1} f' + k L_k(f D), \tag{7.4}$$

where $D = (1/T'(x))'$ is the distortion. We use the formula above to compute symbolical expressions for the derivatives $(L_1 f)^{(l)}$.

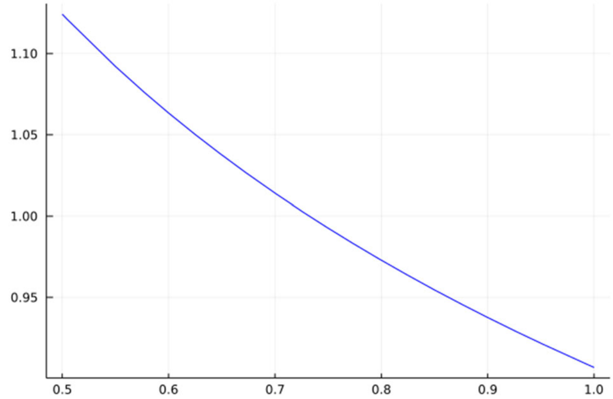


Fig. 4 The invariant density of the induced map as calculated according to Sect. 3.2

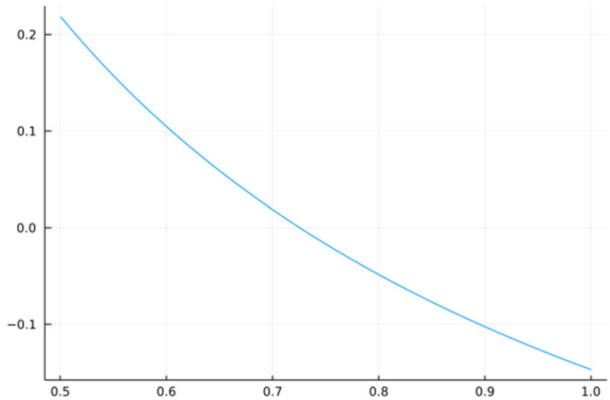


Fig. 5 The linear response of the induced map as calculated according to Sect. 4

We use Interval Arithmetic and higher order Automatic Differentiation [47](as implemented in TaylorSeries.jl) to compute bounds for

$$\left\|(1/T')^{(l)}\right\|_\infty .$$

This allows us to bound the coefficients of the Lasota–Yorke inequalities.

7.3 Approximating the Linear Response for the Induced Map

We approximate the linear response of our induced map using (4.1) which uses the Ulam approximation of $\hat{L}_u = \Pi_u \hat{L}_{\delta_k} \Pi_u$, where Π_u is the Ulam discretisation with partition size $\eta = 9.537 \cdot 10^{-07}$. We get our error from Lemma 4.2, which gives us four terms that need to be bound, each of which is done in Appendix, Sects. 7.3.2, 7.3.3, 7.3.4 and 7.3.7 for $l^* = 99$ and $k = 200$:

(1) $\sum_{n=0}^{l^*} \sum_{i=0}^{n} \left\|(\hat{L}_u - \hat{L}_{\delta_k})\hat{L}_u^{n-i} W_u\right\|_1 \leq 0.0007662$

(2) $\sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\|(\hat{L}_{\delta_k} - \hat{L})\hat{L}_u^{n-i} W_u\right\|_1 \leq 9.616 \cdot 10^{-8}$

(3) $l^* \left\| W_u - W \right\|_1 \leq 3.055 \cdot 10^{-7}$
(4) $\sum_{n=0}^{l^*} \left\| \hat{L}^n W \right\|_1 \leq 2 \cdot 10^{-289}$

this gives us a total error $\left\| \hat{h}^* - \hat{h}_u^* \right\|_1 \leq 0.0007666$

7.3.1 The Contraction Rates of $\hat{L}_{\delta_k}$ in the *BV* Norm

In order to bound $\left\| \hat{L}_{\delta_k}^n |_{U^0} \right\|_{BV}$ we use Lemma 7.13 from [6] to bound

$$\left\| (\hat{L}_{\delta_k}^n - \hat{L}_u^n) f \right\|_1 \leq A_n \left\| f \right\|_{BV} + B_n \left\| f \right\|_1$$

from which we can use $C_{u,n}$ to bound $\left\| \hat{L}_{\delta_k}^n |_{U^0} \right\|_1 \leq A \left\| f \right\|_{BV} + (B + C_{u,n}) \left\| f \right\|_1$ and the Lasota–Yorke inequality (1) from Sect. 8.2.1, and use the small matrix method from [22]. We have

$$\begin{pmatrix} \left\| \hat{L}_{\delta_k}^n f \right\|_{BV} \\ \left\| \hat{L}_{\delta_k}^n f \right\|_1 \end{pmatrix} \leq \begin{pmatrix} \lambda^n & M \\ A_n & B_n + C_{u,n} \end{pmatrix} \begin{pmatrix} \left\| f \right\|_{BV} \\ \left\| f \right\|_1 \end{pmatrix}.$$

We take $C_{u,11} \leq 0.02349$ from Table 1, together with the calculation $\left\| (\hat{L}_{\delta_k}^{11} - \hat{L}_u^{11}) f \right\|_1 \leq 0.002927 \left\| f \right\|_{BV} + 0.0182 \left\| f \right\|_1$, which gives the largest eigenvalue of the small matrix $\rho = 0.09785$.

7.3.2 Bounding Item (1)

We can bound $\sum_{n=0}^{l^*} \sum_{i=0}^{n} \left\| (\hat{L}_u - \hat{L}_{\delta_k}) \hat{L}_u^{n-i} W_u \right\|_1$ by Theorem 7.13 of [6], from (1) we have

$$\left\| \hat{L}_{\delta_k}^n f \right\|_{BV} \leq (0.5)^n \left\| f \right\|_{BV} + 2\,(0.2513 + 1) \left\| f \right\|_1$$

We refer to [22] for the proof of

- $\left\| (\Pi_u - I) f \right\|_1 \leq \eta Var(f) \leq \eta \left\| f \right\|_{BV}$,
- $\left\| \Pi_u \right\|_1 \leq 1$,
- $\left\| \hat{L}_{\delta_k} \right\|_1 \leq 1$.

In the theorem the value of $C_0 = 1$, so we have all of the values we need for the theorem's $N = 1$ case. The theorem then gives

$$\sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| (\hat{L}_u - \hat{L}_{\delta_k}) \hat{L}_u^{n-i} W_u \right\|_1 \leq \eta \frac{3}{2} \sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| \hat{L}_u^{n-i} W_u \right\|_{BV} + \eta \frac{5}{2} M \sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| \hat{L}_u^{n-i} W_u \right\|_1.$$

We can calculate $\left\| \hat{L}_u^{n-i} W_u \right\|_1$ and $\left\| \hat{L}_u^{n-i} W_u \right\|_{BV}$ explicitly by using validated numerical methods, since W_u explicitly represented on the computer, so we can compute an enclosure of

$\hat{L}_u W_u$ by rigorous matrix multiplication; for a function f in the Ulam basis with coefficients v_i we have explicit functions that allow us to compute the L_1, BV and L^∞ norms, i.e.:

$$\|f\|_{L^1} = \eta \sum |v_i|, \quad \mathrm{Var}(f) = \sum |v_{i+1} - v_i|, \quad \|f\|_\infty = \max_i |v_i|.$$

We compute a bound

$$\sum_{n=0}^{l^*} \sum_{i=0}^{n} \left\| (\hat{L}_u - \hat{L}_{\delta_k}) \hat{L}_u^{n-i} W_u \right\|_1 \leq 0.0007662.$$

7.3.3 Bounding Item (2)

To bound $\sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| (\hat{L}_{\delta_k} - \hat{L}) \hat{L}_u^{n-i} W_u \right\|_1$ we observe that as in Lemma 3.3

$$\begin{aligned}
\left\| (L - L_{\delta_k}) f \right\|_1 &= \left\| \sum_{|\omega| > N^*} f \circ g_\omega |g'_\omega| + f(\delta_k(2x-1) + \frac{1}{2}) 2\delta_k \right\|_1 \\
&\leq \sum_{|\omega| > N^*} \| f \circ g_\omega |g'_\omega| \|_1 + \left\| f(\delta_k(2x-1) + \frac{1}{2}) 2\delta_k \right\|_1 \\
&\leq \|f\|_{BV} \left(\sum_{|\omega| > N^*} |g'_\omega| + 2\delta_k \right),
\end{aligned}$$

therefore

$$\sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| (\hat{L}_{\delta_k} - \hat{L}) \hat{L}_u^{n-i} W_u \right\|_1 \leq \left\| (\hat{L}_{\delta_k} - \hat{L}) \right\|_{BV \to L^1} \sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| L_u^{n-i} W_u \right\|_{BV}$$

As in the estimate for item (1) we can compute $\left\| \hat{L}_u^{n-i} W_u \right\|_1$ and $\left\| \hat{L}_u^{n-i} W_u \right\|_{BV}$ explicitly, which gives us

$$\sum_{n=1}^{l^*} \sum_{i=0}^{n-1} \left\| (\hat{L}_{\delta_k} - \hat{L}) \hat{L}_u^{n-i} W_u \right\|_1 \leq 9.616 \cdot 10^{-8}.$$

7.3.4 Bounding Item (3)

We bound $\|W_u - W\|_1$ by the following

$$\|(W_u - W)\|_1 \leq \left\| (W_u - W_{u_\infty}) \right\|_1 + \left\| (W_{u_\infty} - W) \right\|_1$$

where

$$W_{u_\infty} = \sum_\omega \frac{1}{\eta} \int_{g_\omega(I_i)} A_0(x) \cdot \hat{h}'_n(\zeta) + B_0(x) \hat{h}_n(\zeta) dx$$

for $x \in I_i$ and $\zeta \in g_\omega(I_i)$[6] and the sum is over all ω.

$$\begin{aligned}
&\|W - W_{u_\infty}\|_1 \\
&\quad = \left\| \hat{L}[A_0\hat{h}' + B_0\hat{h}](x) - \sum_\omega \frac{1}{|I_i|}\int_{g_\omega(I_i)} A_0(x)\cdot \hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)dx \right\|_{L^1} \\
&\quad \le \int_{I_i}\sum_\omega \left| [A_0\cdot\hat{h}' + B_0\cdot\hat{h}]\circ g_\omega(y)g_\omega'(y) - \frac{1}{|I_i|}\int_{g_\omega(I_i)} A_0(x)\cdot \hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)dx \right| dy \\
&\quad = \sum_\omega \int_{g_\omega(I_i)} \left| [A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z) - \frac{1}{g_\omega'(\hat{T}(z))}\frac{1}{|I_i|}\int_{g_\omega(I_i)} A_0(x)\cdot \hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)dx \right| dz
\end{aligned}$$

where we used the change of variables $z = g_\omega(y)$. The expression above is then equal to

$$\begin{aligned}
&\sum_\omega \int_{g_\omega(I_i)} \left| \int_{g_\omega(I_i)} \frac{1}{|g_\omega(I_i)|}[A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z) - T'(z)\frac{A_0(x)\cdot\hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)}{|I_i|}dx \right| dz \\
&\quad \le \sum_\omega \int_{g_\omega(I_i)}\int_{g_\omega(I_i)} \left| \frac{1}{|g_\omega(I_i)|}[A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z) - \hat{T}'(z)\frac{A_0(x)\cdot\hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)}{|I_i|} \right| dxdz \\
&\quad = \sum_\omega \sum_{*\in\{+,-\}} \int_{g_\omega(I_i)} \left| \int_{g_\omega(I_i)} 1_* \frac{1}{|g_\omega(I_i)|}[A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z) - \hat{T}'(z)\frac{A_0(x)\cdot\hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)}{|I_i|}dz \right| dx
\end{aligned}$$

Here 1_+ is the indicator function on the set where

$$\frac{1}{|g_\omega(I_i)|}[A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z) - \hat{T}'(z)\frac{A_0(x)\cdot\hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)}{|I_i|}$$

is positive, and 1_- is the set on which it is negative.

In the following use $1_{+,+}$ to be the indicator function for the set where the above function is positive and

$$\int_{g_\omega(I_i)} 1_* \frac{1}{|g_\omega(I_i)|}[A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z)dz - A_0(x)\cdot\hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)$$

is positive, $1_{+,-}$ where they are positive and negative, $1_{-,+}$ where they are negative and positive and $1_{-,-}$ to be the indicator of the set on which they are both negative. Then continuing our inequality we have that the expression above is equal to

$$\begin{aligned}
&\sum_\omega \sum_{*,*\in\{+,-\}^2} \left| \int_{g_\omega(I_i)}\int_{g_\omega(I_i)} 1_{*,*}\frac{1}{|g_\omega(I_i)|}[A_0\cdot\hat{h}' + B_0\cdot\hat{h}](z)dz - A_0(x)\cdot\hat{h}_n'(\zeta) + B_0(x)\hat{h}_n(\zeta)dx \right| \\
&= \sum_\omega \sum_{*,*\in\{+,-\}^2} \left| \int_{g_\omega(I_i)} 1_{*,*}A_0(z)\cdot(\hat{h}'(z) - \hat{h}_n'(\zeta))dz \right. \\
&\quad \left. + \int_{g_\omega(I_i)} 1_{*,*}B_0(z)\cdot(\hat{h}(z) - \hat{h}_n(\zeta))dz \right|
\end{aligned}$$

[6] It should be noted that finding the integral of $A_0\hat{h}_n'$ and $B_0\hat{h}_n$ is not easy so in our calculations instead of a true Ulam approximation where ζ corresponds to the value that gives the integral we simply use the midpoint; this error is taken into account explicitly and depends on the regularity estimates we have on h_η.

by rearrangement. We then use Hölder's inequality

$$\leq \sum_{\omega} \sum_{*,*\in\{+,-\}^2} |(|g_\omega(I_i)| \left\|\hat{h}''\right\|_\infty + \left\|\hat{h}-\hat{h}_\eta\right\|_{C^1}) \left|\int_{g_\omega(I_i)} 1_{*,*} A_0(z)dz\right|$$
$$+ (|g_\omega(I_i)| \left\|h'\right\|_\infty + \left\|\hat{h}-\hat{h}_\eta\right\|_{C^1}) \left|\int_{g_\omega(I_i)} 1_{*,*} B_0(z)dz\right|$$
$$\leq |I_i|(\frac{\left\|h''\right\|_\infty \left\|A_0\right\|_1}{2} + \frac{\left\|h'\right\|_\infty \left\|B_0\right\|_1}{2}) + \left\|\hat{h}-\hat{h}_\eta\right\|_{C^1} (\left\|A_0\right\|_1 + \left\|B_0\right\|_1)$$

We can calculate this using a computer to for $\eta = 9.537 \cdot 10^{-07}$ to get

$$\left\|(W - W_{u_\infty})\right\|_1 \leq 3.046 \cdot 10^{-7}.$$

We need now to estimate this for b, the linear branch of $\hat{T}_{\delta_k}$:

$$\sum_{I_i} \left\| \frac{1_{I_i}(x)}{|I_i|} \int_{b^{-1}(I_i)} A_0(y)\hat{h}_n'(\zeta)dy + B_0(y)\hat{h}_n(\zeta)dy - \frac{1_{I_i}(x)}{|I_i|} \sum_{\omega>k} \int_{g_\omega(I_i)} A_0(y)\hat{h}_n'(\zeta) + B_0(y)\hat{h}_n(\zeta)dy \right\|_1$$
$$\leq \sum_{I_i} \int_{b^{-1}(I_i)} |A_0(y)\left\|\hat{h}_n'\right\|_\infty + B_0(y)\left\|\hat{h}_n\right\|_\infty|dy + \sum_{\omega>k} \int_{g_\omega(I_i)} |A_0(y)\left\|\hat{h}_n'\right\|_\infty + B_0(y)\left\|\hat{h}_n\right\|_\infty|dy$$
$$= \int_{0.5}^{g_k(1)} |A_0(y)\left\|\hat{h}_n'\right\|_\infty + B_0(y)\left\|\hat{h}_n\right\|_\infty|dy + \int_{0.5}^{g_k(1)} |A_0(y)\left\|\hat{h}_n'\right\|_\infty + B_0(y)\left\|\hat{h}_n\right\|_\infty|dy$$

where we have used that both $\{g_\omega(I_i)\}_{1\leq i\leq N,\omega>k}$ and $\{b^{-1}(I_i)\}_{1\leq i\leq N}$ form a disjoint cover of $(0.5, g_k(1)]$; then we have that the expression above is equal to

$$2\int_{0.5}^{g_k(1)} |A_0(y)\left\|\hat{h}_n'\right\|_\infty + B_0(y)\left\|\hat{h}_n\right\|_\infty|dy$$
$$\leq 2\left\|\hat{h}_n'\right\|_\infty (g_k(1)-0.5)A_0(g_k(1)) + 2\left\|\hat{h}_n\right\|_\infty \mathrm{Var}(1_{(0.5,g_k(1)]}\cdot A_0).$$

We can make this arbitrarily small by increasing k as much as is needed. We calculate $\mathrm{Var}(1_{(0.5,g_k(1)]}\cdot A_0) = \left\|1_{(0.5,g_k(1)]}\cdot B_0\right\|_1$ and according to Sect. 7.3.8 and $A_0(g_k(1))$ according to Sect. 8.

Taking $k = 200$ gives us $\left\|W_u - W_{u_\infty}\right\|_1 \leq 9.007 \cdot 10^{-10}$. This together with the first bound gives $l^* \left\|W_u - W\right\|_1 \leq 3.055 \cdot 10^{-7}$.

7.3.5 The Contraction Rates of $\hat{L}_{\delta_k}$ in the C^1 Norm

In order to bound $\left\|\hat{L}_{\delta_k}^n|_{U^0}\right\|_{C^1}$ we use Lemma 7.13 from [6] to bound

$$\left\|(\hat{L}_{\delta_k}^m - \hat{L}_n^m)f\right\|_{C^1} \leq A_m \left\|f\right\|_{C^2} + B_m \left\|f\right\|_{C^1}$$

from which we can use $C_{c,m}$ to bound $\left\|\hat{L}_{\delta_k}^m|_{U^0}\right\|_{C^1} \leq A\left\|f\right\|_{C^2} + (B + C_{c,m})\left\|f\right\|_{C^1}$ and the Lasota–Yorke inequality (3) from Sect. 8.2.1, and use the small matrix method from [22]. We have

$$\begin{pmatrix} \left\|\hat{L}_{\delta_k}^m f\right\|_{C^2} \\ \left\|\hat{L}_{\delta_k}^m f\right\|_{C^1} \end{pmatrix} \leq \begin{pmatrix} M\lambda^{2m} & D \\ A_n & B_n + C_{c,m} \end{pmatrix} \begin{pmatrix} \left\|f\right\|_{C^2} \\ \left\|f\right\|_{C^1} \end{pmatrix}.$$

Choosing the value of n that minimises equation (7.3), we take $C_{c,4} \leq 0.1427$ from Sect. 7.2.2, together with the calculation $\left\|(\hat{L}^4_{\delta_k} - \hat{L}^4_n) f\right\|_1 \leq 8.5 \cdot 10^{-7} \|f\|_{C^2} + 3.203 \cdot 10^{-6} \|f\|_{C^1}$, which gives the largest eigenvalue of the small matrix $\rho = 0.1557$.

7.3.6 The Contraction Rates of $\hat{L}$ in the C^1 Norm

We can use $\left\|\hat{L}^n_{\delta_k}|_{U^0}\right\|_{C^1} \leq \rho_{n,C^1}$ and Lemma 3.3 to get

$$\left\|\hat{L}^n|_{U^0}\right\|_{C^1} \leq \left\|\hat{L}^n_{\delta_k}|_{U^0}\right\|_{C^1} + \left\|\hat{L}^n_{\delta_k} - \hat{L}^n\right\|_{C^1} \leq nC^*(2 + D_0 D + D)\delta_k + \rho_{n,C^1}.$$

These give us $\left\|\hat{L}^n|_{U^0}\right\|_{C^1} \leq 2.991 \cdot 10^{-7}$ for $n = 22$.

7.3.7 Bounding Item (4)

From Sect. 7.3.6 we have bounds on $\left\|\hat{L}^n|_{U^0}\right\|_1 \leq C_n$ and we can then write

$$\sum_{n=l^*}^{\infty} \left\|\hat{L}^n W\right\|_1 \leq l^* \frac{C_{l^*} \|W\|_1}{1 - C_{l^*}}.$$

Calculating $\|W\|_1$ gives us

$$\begin{aligned} &\left\|\hat{L}[A_0\hat{h}' + B_0\hat{h}]\right\|_1 \\ &\leq \left\|[A_0\hat{h}' + B_0\hat{h}]\right\|_1 \\ &\leq \|A_0\|_1 \left\|\hat{h}'\right\|_\infty + \|B_0\|_1 \left\|\hat{h}\right\|_\infty \\ &\leq \max\{\|A_0\|_1, \|B_0\|_1\} \left\|\hat{h}\right\|_{C^1} \end{aligned}$$

which we calculate $\|A_0\|_1$ and $\|B_0\|_1$ as described in Sect. 7.3.8, which gives a bound $\|W\|_1 \leq 968.7$. We have from Sect. 7.3.6 that $\left\|\hat{L}^n|_{U^0}\right\|_{C^1} \leq 2.991 \cdot 10^{-7}$, for $n = 22$ so we choose l^* to be a multiple of 22 which gives for $l^* = 99$

$$\sum_{n=l^*}^{\infty} \left\|\hat{L}^n W\right\|_1 \leq 2 \cdot 10^{-289}$$

7.3.8 Calculating $\|A_0\|_1$ and $\|B_0\|_1$

For calculating $\left\|\hat{h}^* - \hat{h}^*_u\right\|_1$ we need bounds on $\|A_0\|_1$ and $\|B_0\|_1$, as used in Sects. 7.3.4 and 7.3.7. We have a method to calculate the values of A_0 and B_0 from Appendix, since $B_0 = A_0'$ we can calculate the integral of $\int_{[a,b]} B_0(x)dx = A_0(b) - A_0(a)$. In order to calculate the integral of A_0 we approximate it by taking $k = 10$ evenly spaced values in each partition element I_i, we then take $\frac{|I_j|}{k} \sum_{j=1}^{k} A_0(x_j)$ as the value of the integral on I_i. This has an L^1 error of $\frac{|I_i|}{k}\text{Var}(A_0)$. Taking our approximation of the integral of A_0 and adding $\frac{|I_i|}{k}\text{Var}(A_0)$ gives an upper bound of $\|A_0\|_1$.

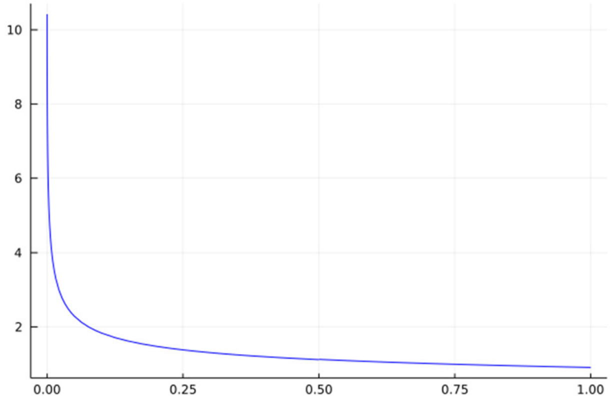


Fig. 6 The invariant density of the LSV map for $\alpha = 0.125$ calculated according to Sect. 2.1 with L^1 error of $7.666 \cdot 10^{-9}$

7.4 Pulling Back to the Original Map

To get the invariant density and linear response for the full map we must pull them back to the unit interval with F and Q from Sect. 2.1. The invariant density is fairly straight forward to calculate and find the error. We want a bound on $\left\| h - F_0^{app} \hat{h}_n \right\|_1$ for which we can use a bound from (I) in the proof of Theorem 2.18.

$$\left\| F_0 \hat{h} - F_0^{app} \hat{h}_n \right\|_1 \leq 2 \left\| \hat{h}_n - \hat{h} \right\|_1 + \frac{1}{2^{1-\gamma}(1-\gamma)} \left\| \hat{h}_n \right\|_{C^0} \sum_{\omega > N^*} \| g'_\omega \|_B$$

We use the bounds for $\gamma = 0.5$ $\frac{1}{2^{1-\gamma}(1-\gamma)} \leq 1.418$, $\sum_{\omega > N^*} \| g'_\omega \|_B \leq 6.51 \cdot 10^{-10}$ as calculated in Sect. 8.1 and $\left\| \hat{h}_n \right\|_{C^0} \leq 1.5$, which gives the second term to be bounded by $1.381 \cdot 10^{-9}$. The first term we can bound by $2 \left\| \hat{h}_n - \hat{h} \right\|_{C^1} \leq 7.666 \cdot 10^{-9}$ as calculated in Sect. 7.2.2, giving us $\left\| h - F_0^{app} \hat{h}_n \right\|_1 \leq 7.666 \cdot 10^{-9}$.

As seen in Theorem 2.18 pulling back the linear response requires the following bounds

$$\begin{aligned} \| h^* - h_u^* \|_1 \leq & \left\| F_0 \hat{h}^* - F_0 \hat{h}_u^* \right\|_1 + \left\| F_0 \hat{h}_u^* - F_0^{app} \hat{h}_u^* \right\|_1 \\ & + \left\| Q \hat{h} - Q \hat{h}_n \right\|_1 + \left\| Q \hat{h}_n - Q^{app} \hat{h}_n \right\|_1 . \end{aligned}$$

We bound these in Sect. 7.4.1 giving

(1) $\left\| F_0 \hat{h}^* - F_0 \hat{h}_u^* \right\|_1 \leq 0.001533$
(2) $\left\| F_0 \hat{h}_u^* - F_0^{app} \hat{h}_u^* \right\|_1 \leq 4.603 \cdot 10^{-10}$
(3) $\left\| Q \hat{h} - Q \hat{h}_n \right\|_1 \leq 6.818 \cdot 10^{-6}$
(4) $\left\| Q \hat{h}_n - Q^{app} \hat{h}_n \right\|_1 \leq 0.006225$

This gives us $\left\| h^* - h_\eta^* \right\|_1 \leq 0.007765$.

7.4.1 Bounding Items (1) and (2)

It is given in Theorem 2.18 that

$$\left\| F_0 \hat{h}^* - F_0 \hat{h}_u^* \right\|_1 \leq 2 \left\| \hat{h}^* - \hat{h}_u^* \right\|_1 ,$$

which we have from Sect. 7.3 is bounded by $2 \cdot 0.0007666 = 0.001533$.

We also have from Theorem 2.18 that $\left\| F_0 \hat{h}_u^* - F_0^{app} \hat{h}_u^* \right\|_1$ is bounded by

$$\left\| \hat{h}_u^* \right\|_{C^0} \frac{1}{2^{1-\gamma}(1-\gamma)} \sum_{\omega > N^*} \| g'_\omega \|_B .$$

We can compute explicitly $\left\| \hat{h}_u^* \right\|_{C^0}$ and it is bounded by 0.5, $\frac{1}{2^{1-\gamma}(1-\gamma)} \leq 1.418$ and for $N^* = 1000$, $\sum_{\omega > N^*} \| g'_\omega \|_B \leq 6.51 \cdot 10^{-10}$ as is shown in Sect. 8.1. These give us the bound $\left\| F_0 \hat{h}_u^* - F_0^{app} \hat{h}_u^* \right\|_1 \leq 4.603 \cdot 10^{-10}$.

7.4.2 Bounding Items (3) and (4)

We can bound $\left\| Q\hat{h} - Q\hat{h}_n \right\|_1$ by

$$\frac{1}{2^{1-\gamma}(1-\gamma)} \sum_{\omega} \| b_\omega \|_B \left\| \hat{h} - \hat{h}_n \right\|_{C^1}$$

for which we will need a bound on $\sum_\omega \| b_\omega \|_B$. In Sect. 8.3 we show is less than 1258; the bounds from earlier $\frac{1}{2^{1-\gamma}(1-\gamma)} \leq 1.418$ and $\left\| \hat{h} - \hat{h}_n \right\|_{C^1} \leq 3.833 \cdot 10^{-9}$ allow us to prove that

$$\left\| Q\hat{h} - Q\hat{h}_n \right\|_1 \leq 6.818 \cdot 10^{-6}.$$

We now bound

$$\begin{aligned} &\left\| Q\hat{h}_n - Q^{app}\hat{h}_n \right\|_1 \\ &\quad \leq \frac{1}{2^{1-\gamma}(1-\gamma)} \left[\sup_\omega |a_\omega| \cdot \left\| \hat{h}'_n \right\|_{C^0} \cdot \sum_{|\omega| > N^*} \| g'_\omega \|_B + \left\| \hat{h}_n \right\|_{C^0} \sum_{\omega > N^*} \| b_\omega \|_B \right]. \end{aligned}$$

We need a bound on $\sum_\omega |a_\omega|$ which we show is bounded by 7107 in Sect. 8.4, and the bounds from computer approximations of $\hat{h}_n$ which gives us $\left\| \hat{h}_n \right\|_{C^0} \leq 1.5$ and $\left\| \hat{h}'_n \right\|_{C^0} \leq 1$.

We can use the same method from Sect. 8.3 to calculate $\sum_{\omega > N^*} \| b_\omega \|_B \leq 0.002931$. All this together gives us

$$\left\| Q\hat{h}_n - Q^{app}\hat{h}_n \right\|_1 \leq 0.006225.$$

7.5 Normalizing the Density and the Linear Response

In this subsection we follow the estimates in Sect. 5.

First of all, we compute

$$\left\| \frac{h_n}{\int h_n dm} - \frac{h}{\int h dm} \right\|_1 \leq 4.11 \cdot 10^{-7}.$$

Following through the calculations we bound (5.1), the L^1 error on the normalized linear response by 0.01501.

8 Effective Bounds for [4, 36]

In this section we will use often the following notation following [36]. Let T_0 be the left branch of the map T, let $z \in [0, 1]$ and

$$z_r := T_0^{-r}(z).$$

By $(.)'$ we denote the derivative with respect to z. To simplify the lookup of constants, they are presented in table 2.

8.1 Estimating the Tail $\sum_{\omega > N^*} \left\| g'_\omega \right\|_B$

For this we look at [4, Lemma 5.2] which gives

$$\left\| g'_\omega \right\|_B \leq C_8 \sup_{z \in (0,0.5]} z^\gamma (1 + n z^\alpha \alpha 2^\alpha)^{-1/\alpha - 1}.$$

using calculations from Sect. 8.3

$$\begin{aligned} &\sup_{z \in (0,0.5]} z^\gamma (1 + n z^\alpha \alpha 2^\alpha)^{-1/\alpha - 1} \\ &\leq \sup_{z \in (0,0.5]} \frac{z^{\gamma - 1 - \alpha} (\alpha 2^\alpha)^{-1/\alpha - 1}}{(z^{-\alpha} \alpha^{-1} 2^{-\alpha} + n)^{1/\alpha + 1}} \\ &\leq C_{sum} n^{-\gamma/\alpha} \end{aligned}$$

so $\sum_{n > N^*} \left\| g'_\omega \right\|_B \leq C_8 \cdot C_{sum} [\zeta(\gamma/\alpha) - \sum_{j=1}^{N^*} j^{-\gamma/\alpha}]$.

The constant C_8 comes from [36] where it is shown to be finite, but, when we calculate C_8 according to their proof we get $C_8 = \exp\left(1 + (\alpha + 1)^2 2^{2\alpha} C_2^2 \frac{\pi^2}{6}\right)$, which is of order 10^{269}.

Therefore we need a sharper bound for C_8. We start similarly

$$\begin{aligned} z'_n &= \Pi_{j=1}^n \frac{1}{1 + (\alpha + 1) 2^\alpha z_j^\alpha} = \exp\left(\sum_{j=1}^n - \log\left(1 + (\alpha + 1) 2^\alpha z_j^\alpha\right) \right) \\ &= \exp\left(\sum_{j=1}^n -(\alpha + 1) 2^\alpha z_j^\alpha + \sum_{j=1}^n [- \log\left(1 + (\alpha + 1) 2^\alpha z_j^\alpha\right) + (\alpha + 1) 2^\alpha z_j^\alpha] \right) \\ &\leq e \cdot (1 + n z_0^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} \cdot \exp \sum_{j=1}^n [- \log\left(1 + (\alpha + 1) 2^\alpha z_j^\alpha\right) + (\alpha + 1) 2^\alpha z_j^\alpha]) \end{aligned}$$

Table 2 Table of constants

Label	Description	Value
α	Parameter for the LSV map	0.125
γ	Parameter for $\|\cdot\|_B$	0.5
δ_k	The size of the interval different from the true induced map	$2.146 \cdot 10^{-12}$
η	Partition size for Ulam discretisation	$9.537 \cdot 10^{-07}$
N^*	The number of branches used to approximate operators F and Q	1000
l^*	The number of iterations used to calculate $\hat{h}_\eta^*$	99
ν	The number of Lasota–Yorke inequalities used to bound error in the Chebyshev projection	5
n	Highest degree of Chebyshev polynomials used for the Chebyshev discretisation	1024
C_1	$\frac{1}{1+\alpha 2^\alpha}$	0.88
C_2	$\frac{1}{\alpha(1-\alpha)2^{\alpha-1}}$	16.77
C_3	$\frac{1}{\alpha} + \log(C_1^{-1/\alpha})$	9.022
C_4	$\frac{1}{\alpha} - \frac{\log(C_1^{1/\alpha})}{\log(2)}$	9.475
C_5	2^α	1.091
C_6	$(\alpha+1)2^\alpha$	1.227
C_7	$\alpha(\alpha+1)2^\alpha$	0.1534
C_8	A computed value from Sect. 8.1	2.766
C_{10}	$C_2 \cdot C_5 \cdot C_8$	50.58
C_{11}	$C_2 \cdot C_6 \cdot C_8 \frac{-\log(1/C_2)+1}{\alpha}$	1739
C_{12}	$2^\alpha C_2^2 \cdot C_4 \cdot C_7 \cdot C_8 \cdot$	1232
C_{sum}	$\frac{(\frac{1+\alpha-\gamma}{\gamma})^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^\alpha)^{-\gamma/\alpha}}{(\frac{1+\alpha-\gamma}{\gamma})^{1/\alpha+1}}$	5.981
D_0	A bound on the distortion of the branches of the induced map	2.956
D	A bound on the distortion of the inverse of the branches of the induced map	19.22

where in the last line we use the calculation in [36] following equation (5.7) which gives,

$$-(\alpha+1)2^\alpha \sum_{j=1}^{n} z_j^\alpha \leq -\frac{\alpha+1}{\alpha}(\log(1+nz_0^\alpha \alpha 2^\alpha) + C$$

where C comes from

$$\sum_{j=1}^{r} \frac{1}{z_0^{-\alpha} + j\alpha 2^\alpha} \geq \int_1^r \frac{z_0^\alpha}{1+tz_0^\alpha \alpha 2^\alpha} dt - C.$$

Since the function in the integral is monotonically decreasing and $\frac{z_0^\alpha}{1+tz_0^\alpha \alpha 2^\alpha} \leq 1$ we can bound C by 1, which gives us the factor of e.

In the next paragraph we will use the Taylor expansion of $-\log(1+x)$, however this is only convergent for $x \in (-1, 1)$, so first we choose a j^* large enough that $\frac{-(\alpha+1)2^\alpha C_2}{j^*} \in (-1, 1)$.

$$\begin{aligned} z'_n &\leq e \cdot (1 + n z_0^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} \cdot \exp \sum_{j=1}^{n} [-\log(1 + (\alpha+1)2^\alpha z_j^\alpha) + (\alpha+1)2^\alpha z_j^\alpha]) \\ &= e \cdot (1 + n z_0^\alpha \alpha 2^\alpha)^{-(\alpha+1)/\alpha} \cdot \exp \sum_{j=1}^{j^*-1} [-\log(1 + (\alpha+1)2^\alpha z_j^\alpha) + (\alpha+1)2^\alpha z_j^\alpha]) \\ &\quad \times \exp \sum_{j=j^*}^{n} [-\log(1 + (\alpha+1)2^\alpha z_j^\alpha) + (\alpha+1)2^\alpha z_j^\alpha]) \end{aligned}$$

For $\exp\left(\sum_{j=j^*}^{n} [-\log(1 + (\alpha+1)2^\alpha z_j^\alpha) + (\alpha+1)2^\alpha z_j^\alpha]\right)$ we use

$$\begin{aligned} &\exp\left(\sum_{j=j^*}^{n} [-\log(1 + (\alpha+1)2^\alpha z_j^\alpha) + (\alpha+1)2^\alpha z_j^\alpha]\right) \\ &\leq \exp\left(\sum_{j=j^*}^{n} [-\log(1 + (\alpha+1)2^\alpha C_2 j^{-1}) + (\alpha+1)2^\alpha C_2 j^{-1}]\right). \end{aligned}$$

Substituting in the Taylor expansion of $\log(1+x)$ where $x = (\alpha+1)2^\alpha C_2 j^{-1}$ gives

$$\begin{aligned} &\exp\left(\sum_{j=j^*}^{n} \sum_{m=2}^{\infty} \frac{(-(\alpha+1)2^\alpha C_2)^m}{m} j^{-m}\right) \\ &= \exp\left(\sum_{m=2}^{\infty} \sum_{j=j^*}^{n} \frac{(-(\alpha+1)2^\alpha C_2)^m}{m} j^{-m}\right) \\ &= \exp\left(\sum_{m=2}^{\infty} \frac{(-(\alpha+1)2^\alpha C_2)^m}{m} \sum_{j=j^*}^{n} j^{-m}\right) \\ &\leq \exp\left(\sum_{m=2}^{\infty} \frac{(-(\alpha+1)2^\alpha C_2)^m}{m} [\zeta(m) - \sum_{j=1}^{j^*} \frac{1}{j^m}]\right) \\ &\leq \exp\left(\sum_{m=2}^{\infty} \frac{(-(\alpha+1)2^\alpha C_2)^m}{m} [\zeta(2) - \sum_{j=1}^{j^*} \frac{1}{j^2}]\right) \\ &\leq \exp(-[\zeta(2) - \sum_{j=1}^{j^*} \frac{1}{j^2}] \cdot [\log(1 + (\alpha+1)2^\alpha C_2) - (\alpha+1)2^\alpha C_2]) \\ &= \exp((\alpha+1)2^\alpha C_2 [\zeta(2) - \sum_{j=1}^{j^*} \frac{1}{j^2}]) \cdot (1 + (\alpha+1)2^\alpha C_2)^{-[\zeta(2) - \sum_{j=1}^{j^*} \frac{1}{j^2}]} \end{aligned}$$

To get our final estimate we need to bound

$$\exp \sum_{j=1}^{j^*-1} [-\log(1+(\alpha+1)2^\alpha z_j^\alpha) + (\alpha+1)2^\alpha z_j^\alpha]),$$

Since there are a finite number of terms we can bound it from above through the use of rigorous numerical methods.

Choosing $j^* = 586$ gives us that $\frac{-(\alpha+1)2^\alpha C_2}{j^*} = -0.02057 \in (-1, 1)$ and $C_8 \geq 2.766$ is an upper bound.

This gives us that for $N^* = 1000$, $\sum_{\omega > N^*} \left\| g'_\omega \right\|_B \leq 6.51 \cdot 10^{-10}$.

8.2 Bounds for Lemma 3.3

We want a bound on $D_0 = \left\| \frac{g''_\omega}{g'_\omega} \right\|_\infty$. In [36] they have bounds for z_r where $g'_{r+1} = 0.5 z'_r$ and $g''_{r+1} = 0.5 z''_r$ so we may use the bound from [36, Lemma 5.4] which gives

$$\frac{z''_r}{z'_r} = \frac{\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1} z'_{r+1}}{1+(\alpha+1)2^\alpha z_{r+1}^\alpha} + \frac{z''_{r+1}}{z'_{r+1}}$$

which we can use to get a bound on $\frac{z''_r}{z'_r} - \frac{z''_{r+1}}{z'_{r+1}}$. We use Lemmas 5.2 and 5.3 from [36] to get

$$z'_r \leq C_8(\alpha 2^\alpha)^{-(\alpha+1)/\alpha} r^{-(\alpha+1)/\alpha} z_0^{-\alpha-1}$$

where we calculate C_8 in Sect. 8.1, and

$$z_r^{\alpha-1} = (z_r^\alpha)^{(\alpha-1)/\alpha} \leq \left(\frac{2^{1-\alpha}}{\alpha(1-\alpha)} \right)^{(\alpha-1)/\alpha} r^{-(\alpha-1)/\alpha}$$

giving us

$$z_r^{\alpha-1} z'_r \leq C_8 \frac{\alpha^{-2} 2^{1-1/\alpha-2\alpha}}{(1-\alpha)^{(\alpha-1)/\alpha}} r^{-2} z_0^{-\alpha-1}.$$

Then

$$\begin{aligned}\frac{z''_r}{z'_r} - \frac{z''_{r+1}}{z'_{r+1}} &\leq \alpha(\alpha+1)2^\alpha C_8 \frac{\alpha^{-2} 2^{1-1/\alpha-2\alpha}}{(1-\alpha)^{(\alpha-1)/\alpha}} r^{-2} z_0^{-\alpha-1} \\ &\leq (\alpha+1) C_8 \frac{\alpha^{-1} 2^{1-1/\alpha-\alpha}}{(1-\alpha)^{(\alpha-1)/\alpha}} r^{-2} 0.5^{-\alpha-1}\end{aligned}$$

from which follows that

$$sup_r \left\| \frac{z''_r}{z'_r} \right\|_\infty \leq \pi^2 \frac{0.5^{-\alpha-1}}{6} (\alpha+1) C_8 \frac{\alpha^{-1} 2^{1-1/\alpha-\alpha}}{(1-\alpha)^{(\alpha-1)/\alpha}}$$

which for $\alpha = 0.125$ gives $D_0 = 2.956$.

Since

$$\begin{aligned}\log\left(\frac{g'_\omega(x)}{g'_\omega(y)}\right) &= [\log(g'_\omega(\zeta))]'(x-y) \\ &= \frac{g''_\omega(\zeta)}{g'_\omega(\zeta)}(x-y) \leq \frac{g''_\omega(\zeta)}{g'_\omega(\zeta)},\end{aligned}$$

we have that $D \leq \exp(D_0) \leq 1.286$.

8.2.1 Lasota–Yorke Inequalitys for $\hat{L}_{\delta_k}$ and $\hat{L}$.

We use some estimates from [6];

(1) From Proposition 7.2 we have $var(\hat{L}f) \leq \lambda var(f) + B\|f\|_1$ with $B := \left\|\hat{T}''/(\hat{T}')^2\right\|_\infty$ and $\lambda := 1/\inf_x |D_x\hat{T}|$;
(2) From Proposition 7.4 $\left\|\hat{L}^n f\right\|_{C^1} \leq M\lambda^n \|f\|_{C^1} + M^2\|f\|_\infty$ with $M := 1 + \frac{B}{1-\lambda}$;
(3) From Proposition 7.6 $\left\|\hat{L}^n f\right\|_{C^2} \leq M(\lambda^2)^n\|f\|_{C^2} + D\|f\|_{C^1}$ where $D :=$ max $\left\{3\frac{\lambda BM}{1-\lambda}, 3\,M\left(\frac{B}{1-\lambda}\right)^2 + MZ\right\} + M\lambda + M^2$. Z being

$$\frac{1}{1-\lambda^2}\left(\left\|\hat{T}'''/(\hat{T}')^3\right\|_\infty + \frac{3\lambda}{1-\lambda}\left\|\hat{T}''/(\hat{T}')^2\right\|_\infty\right).$$

By the construction of $\hat{T}_{\delta_k}$ we know $1/\inf_x|D_x\hat{T}_{\delta_k}| = 1/\inf_x|D_x\hat{T}|$, $\left\|\hat{T}''_{\delta_k}/(\hat{T}'_{\delta_k})^2\right\|_\infty \leq \left\|\hat{T}''/(\hat{T}')^2\right\|_\infty$ and $\left\|\hat{T}'''_{\delta_k}/(\hat{T}'_{\delta_k})^3\right\|_\infty \leq \left\|\hat{T}'''/(\hat{T}')^3\right\|_\infty$ so bounding these values for $\hat{T}$ gives us inequalities that are true for both.

We note that $\left\|\hat{T}''/(\hat{T}')^2\right\|_\infty = \sup_\omega \left\|g''_\omega/(g'_\omega)\right\|_\infty = D_0$ which is calculated in Sect. 8.2. We can calculate $\left\|\hat{T}'''/(\hat{T}')^3\right\|_\infty$ a similar way as follows,

$$\left\|\frac{T'''}{(T')^3}\right\|_\infty = \left\|\left(\frac{g'''_\omega}{(g'_\omega)^4} + 3\frac{(g''_\omega)^2}{(g'_\omega)^5}\right)(g'_\omega)^3\right\|_\infty \leq \left\|\frac{g'''_\omega}{g'_\omega}\right\|_\infty + 3\left\|\left(\frac{g''_\omega}{g'_\omega}\right)^2\right\|_\infty$$

so we need to bound $\left\|\frac{g'''_\omega}{g'_\omega}\right\|_\infty$ and $\left\|\left(\frac{g''_\omega}{g'_\omega}\right)^2\right\|_\infty$, the second of which is D_0^2 from Sect. 8.2. In [36] it is proven that $\left\|\frac{g'''_\omega}{g'_\omega}\right\|_\infty$ is bounded and their method gives that it is less than

$$\sum_{r=0}^{\infty}(\alpha-1)\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-2}(z'_{r+1})^2 + 3\alpha(\alpha+1)2^\alpha z_{r+1}^{\alpha-1}z'_{r+1}\frac{z''_{r+1}}{z'_{r+1}}$$

where $g_\omega = z_r \circ g_1$, so $z'_r = \frac{g'_\omega}{2}$ and $z'''_r = \frac{g'''_\omega}{2}$. We have bounds on $\frac{z''_{r+1}}{z'_{r+1}}$ and $z_{r+1}^{\alpha-1}z'_{r+1}$ from Sect. 8.2, and we bound $z_{r+1}^{\alpha-2}$ using Lemma 5.2 of [36] to get $z_{r+1}^{\alpha-2} \leq (\frac{C_2}{r+1})^{(\alpha-2)/\alpha}$. We bound $(z'_{r+1})^2$ using Lemma 5.3 of [36] to get $(z'_{r+1})^2 \leq C_8^2(2^\alpha\alpha)^{-2(\alpha+1)/\alpha}z_0^{-2(\alpha+1)}(r+1)^{-2(\alpha+1)/\alpha}$ so we can bound $\left\|\frac{g'''_\omega}{g'_\omega}\right\|_\infty$ by

$$\left(z_0^{-\alpha-4}\sum_{r=0}^{\infty}(r+1)^{-3}\right)[(\alpha-1)\alpha(\alpha+1)2^\alpha C_2^{(\alpha-2)/\alpha}C_8^2(2^\alpha\alpha)^{-2(\alpha+1)/\alpha} + 3\alpha(\alpha+1)2^\alpha C]$$

where C is the product of the values from section 8.2. Substituting in the maximizing value of $z_0 = 0.5$ and note $\sum_{r=0}^{\infty}(r+1)^{-3} = \zeta(3)$ to get a bound of 0.08016.

This gives us a bound of $\left\|\frac{T'''}{(T')^3}\right\|_\infty \leq 0.2696$ and we have

- $\lambda = 0.5$,
- $B \leq 0.2513$,
- $M \leq 1.503$,
- $Z \leq 1.365$,
- $D \leq 12.19$.

These values give us the explicit bounds

(1) $\mathrm{Var}(\hat{L}_{\delta_k} f) \leq 0.5\mathrm{Var}(f) + 0.2513 \|f\|_1$;
(2) $\left\|\hat{L}^n_{\delta_k} f\right\|_{C^1} \leq 1.503 \cdot 0.5^n \|f\|_{C^1} + 2.258 \|f\|_\infty$;
(3) $\left\|\hat{L}^n_{\delta_k} f\right\|_{C^2} \leq 1.503 \cdot 0.25^n \|f\|_{C^2} + 12.19 \|f\|_{C^1}$.

These Lasota–Yorke inequalities give us the bounds $C^* = M\lambda + M^2 = 7.452$ for Lemma 3.6.

For a bound on $\left\|\hat{h}_{\delta_k}\right\|_{C^1}$ and $\left\|\hat{h}_{\delta_k}\right\|_{C^1}$ we observe that $\left\|\hat{h}_{\delta_k}\right\|_1 = 1$, and $\hat{L}_{\delta_k}\hat{h}_{\delta_k} = \hat{h}_{\delta_k}$, the inequalities above give us

(1) $var(\hat{h}_{\delta_k}) \leq 0.2513 \implies \left\|\hat{h}_{\delta_k}\right\|_{BV} \leq 1 + 0.2513 = 1.251$
(2) $\left\|\hat{h}_{\delta_k}\right\|_{C^1} \leq 2.258 \left\|\hat{h}_{\delta_k}\right\|_\infty \leq 2.258 \left\|\hat{h}_{\delta_k}\right\|_{BV} = 2.825$
(3) $\left\|\hat{h}_{\delta_k}\right\|_{C^2} \leq 12.19 \left\|\hat{h}_{\delta_k}\right\|_{C^1} \leq 34.45$

8.3 Bounding $\sum_\omega \left\|\partial_\epsilon g'_\omega\right\|_B$

For this we use from Lemma 5.2 [4].

$$\sum_\omega \left\|\partial_\epsilon g'_\omega\right\|_B \leq C_5 \sum_{n=1}^{\infty} \sup_{\alpha\in U} \sup_{z\in(0,0.5]} |z^\gamma \cdot z'_n| \sum_{j=1}^{n} z_j^\alpha$$
$$+ C_6 \sum_{n=1}^{\infty} \sup_{\alpha\in U} \sup_{z\in(0,0.5]} |z^\gamma \cdot z'_n| \sum_{j=1}^{n} z_j^\alpha |\log z_j|$$
$$+ C_7 \sum_{n=1}^{\infty} \sup_{\alpha\in U} \sup_{z\in(0,0.5]} |z^\gamma \cdot z'_n| \sum_{j=1}^{n} z_j^{\alpha-1} |\partial_\alpha z_j|$$

where $C_5 = 2^\alpha$, $C_6 = (\alpha+1)2^\alpha$ and $C_7 = \alpha(\alpha+1)2^\alpha$. We now use our bound from 8.1 get C_8 for $z'_n \leq C_8(1 + rz_0^\alpha 2^\alpha)^{-\frac{\alpha+1}{\alpha}}$. We use this to bound $|z^\gamma \cdot z'_n|$ by $C_8 z^\gamma (1 + nz_0^\alpha 2^\alpha)^{-\frac{\alpha+1}{\alpha}}$.

We then use $z_n^\alpha < \frac{C_2}{n}$ to get $z_j^\alpha \leq C_2 j^{-1}$, $\log(z_j) \leq \frac{-\log(1/C_2)+1}{\alpha} \log j$. Using the fact that $z_n^{\alpha-1}|\partial_\alpha z_n| = z_n^\alpha \frac{|\partial_\alpha z_n|}{z_n}$ and inequality (5.9) from [36]

$$z_n^{\alpha-1}|\partial_\alpha z_n| \leq z_n^\alpha \sum_{j=1}^{n} 2^\alpha z_j^\alpha(-\log(2z_j)) \leq 2^\alpha \frac{C_2^2}{n} \sum_{j=1}^{n} j^{-1}(-\log(2z_j))$$

We now use the following

$$-\log(2z_j) \leq C_4 \log(j)$$

To get a value on C_4 we do the following,

$$\begin{aligned}\frac{-\log(2z_j)}{\log(j)} &\leq \frac{-\log(2(\frac{C_1}{j})^{1/\alpha}z_0)}{\log(j)} \leq \frac{-\log(2C_1^{\frac{1}{\alpha}}z_0)+\frac{1}{\alpha}\log(j)}{\log(j)}\\ &\leq \frac{1}{\alpha}-\frac{\log(2C_1^{\frac{1}{\alpha}}z_0)}{\log(j)} \leq \frac{1}{\alpha}-\frac{\log(2C_1^{\frac{1}{\alpha}}z_0)}{\log(2)}\\ &= C_4 \leq 9.475\end{aligned}$$

where we note that $\partial_\alpha z_j = 0$ for $|\omega| = 1$ and so $j = 1$ this is still a valid bound for $z_j^{\alpha-1}|\partial_\alpha z_j|$ which gives us $\sum_\omega \|b_\omega\|_B \leq i + ii + iii$ where

$$\text{(i)} \leq C_{10}\sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]} z^\gamma(1+nz^\alpha\alpha 2^\alpha)^{-1/\alpha-1}\sum_{j=1}^{n}(j^{-1})$$

$$\text{(ii)} \leq C_{11}\sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]} z^\gamma(1+nz^\alpha\alpha 2^\alpha)^{-1/\alpha-1}\sum_{j=1}^{n}(j^{-1}\log j)$$

$$\text{(iii)} \leq C_{12}\sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]} z^\gamma(1+nz^\alpha\alpha 2^\alpha)^{-1/\alpha-1}\sum_{j=1}^{n}(j^{-1}\sum_{k=1}^{j}k^{-1}\log k)$$

where $C_{10} = C_5 \cdot C_8 \cdot C_2$, $C_{11} = C_6 \cdot C_8 \cdot C_2 \cdot \frac{-\log(1/C_2)+1}{\alpha}$ and $C_{12} = 2^\alpha C_4 \cdot C_7 \cdot C_8 \cdot C_2^2$.

To bound (i) we use that $\sum_{j=1}^{n} j^{-1} \leq 1 + \log(n)$ to get

$$\begin{aligned}z^\gamma(1+nz^\alpha\alpha 2^\alpha)^{-1/\alpha-1}\sum_{j=1}^{n}(j^{-1}) &\leq \frac{z^\gamma(1+\log(n))}{(1+nz^\alpha\alpha 2^\alpha)^{1/\alpha+1}}\\ &\leq \frac{z^{\gamma-\alpha(1/\alpha+1)}(\alpha 2^\alpha)^{-1/\alpha-1}(1+\log(n))}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}} \leq \frac{z^{\gamma-1-\alpha}(\alpha 2^\alpha)^{-1/\alpha-1}(1+\log(n))}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}.\end{aligned}$$

We can use this to bound

$$\begin{aligned}&\sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]} z^\gamma(1+nz^\alpha\alpha 2^\alpha)^{-1/\alpha-1}(1+\log(n))\\ &\quad\leq \sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]}\frac{z^{\gamma-1-\alpha}(\alpha 2^\alpha)^{-1/\alpha-1}(1+\log(n))}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}\\ &\quad= \sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]}\frac{z^{\gamma-1-\alpha}(\alpha 2^\alpha)^{-1/\alpha-1}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}} + \frac{z^{\gamma-1-\alpha}(\alpha 2^\alpha)^{-1/\alpha-1}\log(n)}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}\end{aligned}$$

In order to calculate bounds we must find the $z \in (0, 0.5]$ that gives us the maximum value, which we do by finding the zero of the derivative of the part that depends on z,

$$\begin{aligned}&\partial_z\frac{z^{\gamma-1-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}\\ &\quad= \frac{\partial_z z^{\gamma-1-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}} + z^{\gamma-1-\alpha}\partial_z\frac{1}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}.\end{aligned}$$

Calculating the derivatives and rearranging gives,

$$
\begin{aligned}
&= \frac{(\gamma-1-\alpha)z^{\gamma-2-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}} + z^{\gamma-1-\alpha}\frac{-(1/\alpha+1)\cdot-\alpha\cdot z^{-\alpha-1}\cdot\alpha^{-1}2^{-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+2}}\\
&= \frac{(\gamma-1-\alpha)z^{\gamma-2-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}} + \frac{(1+\alpha)\cdot z^{\gamma-2\alpha-2}\cdot\alpha^{-1}2^{-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+2}}\\
&= \frac{z^{\gamma-\alpha-2}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}\left((\gamma-1-\alpha)+\frac{(1+\alpha)z^{-\alpha}\alpha^{-1}2^{-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)}\right)
\end{aligned}
$$

which is zero when $-(\gamma-1-\alpha)=\frac{(1+\alpha)z^{-\alpha}\alpha^{-1}2^{-\alpha}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)}$, we let $y=z^{-\alpha}\alpha^{-1}2^{-\alpha}$ which gives

$$
\begin{aligned}
&-(\gamma-1-\alpha)=(1+\alpha)\frac{y}{y+n}\\
\implies &-(\gamma-1-\alpha)n+(1+\alpha)y-\gamma y=(1+\alpha)y\\
\implies &-(\gamma-1-\alpha)n-\gamma y=0
\end{aligned}
$$

Therefore $y=\frac{(1+\alpha-\gamma)}{\gamma}n$ and

$$
z=\left(\alpha 2^{\alpha}\frac{(1+\alpha-\gamma)}{\gamma}n\right)^{-1/\alpha}.
$$

We substitute this into the first sum

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sup_{\alpha\in U}\sup_{z\in(0,0.5]}\frac{z^{\gamma-1-\alpha}(\alpha 2^{\alpha})^{-1/\alpha-1}}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}}\\
&=\sum_{n=1}^{\infty}\sup_{\alpha\in U}\frac{(\alpha 2^{\alpha}\frac{(1+\alpha-\gamma)}{\gamma}n)^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^{\alpha})^{-1/\alpha-1}}{(\alpha 2^{\alpha}\frac{(1+\alpha-\gamma)}{\gamma}n\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}},
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sup_{\alpha\in U}\frac{\left(\frac{(1+\alpha-\gamma)}{\gamma}\right)^{-\gamma/\alpha+1/\alpha+1} n^{-\gamma/\alpha}(\alpha 2^{\alpha})^{-\gamma/\alpha}}{\left(\frac{1+\alpha-\gamma}{\gamma}+1\right)^{1/\alpha+1}}\\
&=\sup_{\alpha\in U}\frac{\left(\frac{(1+\alpha-\gamma)}{\gamma}\right)^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^{\alpha})^{-\gamma/\alpha}}{\left(\frac{1+\alpha-\gamma}{\gamma}+1\right)^{1/\alpha+1}}\sum_{n=1}^{\infty}n^{-\gamma/\alpha}\\
&=\sup_{\alpha\in U}\frac{\left(\frac{(1+\alpha-\gamma)}{\gamma}\right)^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^{\alpha})^{-\gamma/\alpha}}{\left(\frac{1+\alpha-\gamma}{\gamma}+1\right)^{1/\alpha+1}}\zeta(\gamma/\alpha).
\end{aligned}
$$

By the same calculation we have the second sum is bounded by

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sup_{\alpha\in U} \sup_{z\in(0,0.5]} \frac{z^{\gamma-1-\alpha}(\alpha 2^{\alpha})^{-1/\alpha-1}\log(n)}{(z^{-\alpha}\alpha^{-1}2^{-\alpha}+n)^{1/\alpha+1}} \\
&\leq \sup_{\alpha\in U} \frac{\left(\frac{(1+\alpha-\gamma)}{\gamma}\right)^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^{\alpha})^{-\gamma/\alpha}}{\left(\frac{1+\alpha-\gamma}{\gamma}+1\right)^{1/\alpha+1}} \sum_{n=1}^{\infty} n^{-\gamma/\alpha}\log(n) \\
&= \sup_{\alpha\in U} \frac{\left(\frac{(1+\alpha-\gamma)}{\gamma}\right)^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^{\alpha})^{-\gamma/\alpha}}{\left(\frac{1+\alpha-\gamma}{\gamma}+1\right)^{1/\alpha+1}} |\zeta'(\gamma/\alpha)|
\end{aligned}$$

which gives us

$$(I) \leq C_{10} \cdot C_{sum}(\zeta(\gamma/\alpha) + |\zeta'(\gamma/\alpha)|).$$

where $C_{sum} = \sup_{\alpha\in U} \frac{(\frac{(1+\alpha-\gamma)}{\gamma})^{-\gamma/\alpha+1/\alpha+1}(\alpha 2^{\alpha})^{-\gamma/\alpha}}{(\frac{1+\alpha-\gamma}{\gamma}+1)^{1/\alpha+1}} \leq 5.981.$

To bound (ii) we do the same, but using $\sum_{j=1}^{n} \log(j) j^{-1} \leq \log^2(n)$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sup_{\alpha\in U} \sup_{z\in(0,0.5]} z^{\gamma}(1+nz^{\alpha}\alpha 2^{\alpha})^{-1/\alpha-1} \sum_{j=1}^{n}(j^{-1}\log j) \\
&\leq C_{sum} \sum_{n=1}^{\infty} n^{-\gamma/\alpha}\log^2(n) \\
&\leq C_{sum}\zeta''(\gamma/\alpha)
\end{aligned}$$

giving

$$\text{(ii)} \leq C_{11} \cdot C_{sum}\zeta''(\gamma/\alpha).$$

For (iii) we use $\sum_{j=1}^{n}(j^{-1}\sum_{k=1}^{j} k^{-1}\log k) \leq \sum_{j=1}^{n}(j^{-1}\log^2(j)) \leq \log^3(n)$ to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sup_{\alpha\in U} \sup_{z\in(0,0.5]} z^{\gamma}(1+nz^{\alpha}\alpha 2^{\alpha})^{-1/\alpha-1} \sum_{j=1}^{n}(j^{-1}\sum_{k=1}^{j} k^{-1}\log k) \\
&\leq C_{sum} \sum_{n=1}^{\infty} n^{-\gamma/\alpha}\log^3(n) \\
&\leq C_{sum}|\zeta'''(\gamma/\alpha)|
\end{aligned}$$

This implies directly that

$$\text{(iii)} \leq C_{12} \cdot C_{sum}\zeta'''(\gamma/\alpha).$$

In order to get the bound closer, we can use the technique of calculating the first N terms of $\sum_{n=1}^{\infty} \|b_{\omega}\|_B$ using the computer calculations from 9 and the range estimation method from [47].

$$\begin{aligned}\sum_{\omega}\|b_\omega\|_B \leq C_{10}\cdot C_{sum}(\zeta(\gamma/\alpha)+|\zeta'(\gamma/\alpha)|-\sum_{j=1}^{N}[j^{-\gamma/\alpha}+j^{-\gamma/\alpha}\log(j)])\\ +C_{11}\cdot C_{sum}(\zeta''(\gamma/\alpha)-\sum_{j=1}^{N}[j^{-\gamma/\alpha}\log^2(j)])\\ +C_{12}\cdot C_{sum}(|\zeta'''(\gamma/\alpha)|-\sum_{j=1}^{N}[j^{-\gamma/\alpha}\log^3(j)])\\ +\sum_{1\leq|\omega|\leq N}\|b_\omega\|_B\,.\end{aligned}$$

We calculate upper bounds on the derivatives of $\zeta(x)$ using methods from [16].

Choosing $N = 1000$ and $j^* = 586$ gives us $\sum_\omega \|b_\omega\|_B \leq 1258$. The tail of the sum starting at $n = 1000$ gives $\sum_{|\omega|\geq n}\|b_\omega\|_B \leq 0.002931$.

8.4 Bounding $\sup_\omega |a_\omega|$

For $\sup_\omega |a_\omega|$ we use Lemma 5.2 from [36]. The proof of this lemma gives us

$$\frac{z_0^\alpha}{n}\cdot\frac{1}{1+\alpha 2^\alpha}\leq z_n^\alpha \leq \frac{1}{z_0^{-\alpha}+n\alpha(1-\alpha)2^{\alpha-1}}$$

from which we get $C_1 = \frac{1}{1+\alpha 2^\alpha}$ and $C_2 = \frac{1}{\alpha(1-\alpha)2^{\alpha-1}}$

$$\frac{z_0^\alpha}{n}\cdot C_1 \leq z_n^\alpha \leq \frac{1}{n}\cdot C_2. \tag{8.1}$$

Then $z_0^\alpha \frac{C_1}{n} \leq z_n^\alpha$ gives us $-\log z_0 \leq \frac{-1}{\alpha}\log\frac{C_1}{n} - \log z_0$. To get a C_3 such that $\frac{-1}{\alpha}\log\frac{C_1}{n} \leq C_3 \operatorname{logg}(n)$ we take $C_3 = \log(C_1^{-1/\alpha}) + \frac{1}{\alpha}$. Since $C_3 > 1$

$$-\log z_0 \leq C_3(\operatorname{logg}(n) - \log z_0).$$

Then from the proof of Lemma 5.2 from [4] we have

$$\partial_\alpha z_{n+1} \leq 2^\alpha \sum_{j=1}^{n+1} z_j^{\alpha+1}(-\log 2z_j) \tag{8.2}$$

where $\sup_\omega |a_\omega| \leq \sup_{z_0\in[0,0.5]} \partial_\alpha z_{n+1}$. We use the fact that $x^{\alpha+1}(-\log 2x)$ is monotonicly increasing below $x = 0.5\exp(\frac{-1}{\alpha+1})$ to say that if $C_2^{1/\alpha} j^{*-1/\alpha} \leq 0.5\exp(\frac{-1}{\alpha+1})$ then

$$\begin{aligned}\partial_\alpha z_{n+1} &\leq 2^\alpha \sum_{j=1}^{n+1} z_j^{\alpha+1}(-\log 2z_j)\\ &= 2^\alpha \sum_{j=j^*}^{n+1} z_j^{\alpha+1}(-\log 2z_j) + 2^\alpha \sum_{j=1}^{j^*-1} z_j^{\alpha+1}(-\log 2z_j).\end{aligned}$$

We may use a computer to calculate the sum up to $j^* - 1$ and we bound the rest as follows,

$$
\begin{aligned}
&2^\alpha \sum_{j=j^*}^{n+1} z_j^{\alpha+1}(-\log 2z_j) \\
&\leq 2^\alpha \sum_{j=j^*}^{n+1} [C_2^{1/\alpha} j^{-1/\alpha}]^{\alpha+1}(-\log(2C_2^{1/\alpha} j^{-1/\alpha})) \\
&= 2^\alpha \sum_{j=j^*}^{n+1} [C_2^{1/\alpha} j^{-1/\alpha}]^{\alpha+1}(-\log(j^{-1/\alpha}) - \log(2C_2^{1/\alpha})) \\
&\leq 2^\alpha \sum_{j=j^*}^{n+1} [C_2^{1/\alpha} j^{-1/\alpha}]^{\alpha+1}(-\log(j^{-1/\alpha})) \\
&\leq \frac{2^\alpha C_2^{(\alpha+1)/\alpha}}{\alpha} \sum_{j=j^*}^{n+1} j^{-1-1/\alpha} \log j.
\end{aligned}
$$

Noticing that $\sum_{j=1}^{\infty} j^{-1-1/\alpha} \log j = -\zeta'(1 + 1/\alpha)$ which can be calculated by methods from [16] gives us

$$
\sup_\omega |a_\omega| \leq \frac{[-\zeta'(1+1/\alpha) - \sum_{j=1}^{j^*-1} j^{-1-1/\alpha} \log j]2^\alpha}{\alpha(\alpha(1-\alpha)2^{\alpha-1})^{(\alpha+1)/\alpha}} + 2^\alpha \sum_{j=1}^{j^*-1} z_j^{\alpha+1}(-\log 2z_j) \tag{8.3}
$$

which for $\alpha = 0.125$ and taking $j^* = 586$ gives $\sup_\omega |a_\omega| \leq 7107$

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Data Availability The code implimented for this work will be published online as soon as possible and is available upon request.

Declarations

Conflict of interest The authors declared that they have no affiliations or involvement with any organization or entity with any financial interest in the subject matter or materials discussed in this work.

Appendix A: Computing Derivatives

In order to calculate A_0, B_0, a_ω and b_ω we use an iterative formula. We start with

$$g_\omega \circ T_\omega(x) = x$$

from which we use the chain rule to get

$$(g_\omega \circ T_\omega)'(x) = g'_\omega \circ T_\omega(x) \cdot T'_\omega(x) = 1$$
$$\implies g'_\omega(x) = \frac{1}{T'_\omega \circ g_\omega(x)}$$

and

$$\partial_\alpha(g_\omega \circ T_\omega)(x) = \partial_\alpha g_\omega \circ T_\omega(x) + g'_\omega \circ T_\omega(x) \cdot \partial_\alpha T_\omega(x) = 0$$
$$\implies \partial_\alpha g_\omega(x) = -\frac{\partial_\alpha T_\omega \circ g_\omega(x)}{T'_\omega \circ g_\omega(x)}.$$

We then use further chain rules to get

$$(g'_\omega \circ T_\omega)'(x) = g''_\omega \circ T_\omega(x) \cdot T'_\omega(x) = -\frac{T''_\omega(x)}{(T'_\omega(x))^2}$$
$$\implies g''_\omega(x) = -\frac{T''_\omega \circ g_\omega(x)}{(T'_\omega \circ g_\omega(x))^3}$$

and

$$\partial_\alpha(g'_\omega \circ T_\omega)(x) = \partial_\alpha g'_\omega \circ T_\omega(x) + g''_\omega \circ T_\omega(x) \cdot \partial_\alpha T_\omega(x) = \partial_\alpha \frac{1}{T'_\omega(x)}$$
$$\implies \partial_\alpha g'_\omega(x) = \frac{T''_\omega \circ g_\omega(x) \cdot \partial_\alpha T_\omega \circ g_\omega(x)}{(T'_\omega \circ g_\omega)^3} - \frac{\partial_\alpha T'_\omega \circ g_\omega(x)}{(T'_\omega \circ g_\omega(x))^2}.$$

We already can calculate g_ω so we need to calculate $\partial_\alpha T_\omega$, T'_ω, $\partial_\alpha T'_\omega$ and T''_ω. Note that $T_\omega = T_0^n \circ T_1$ where $|\omega| = n$, so

$$\begin{aligned}
(T_0^n \circ T_1)' &= (T_0^n)' \circ T_1 \cdot T_1' \\
\partial_\alpha(T_0^n \circ T_1) &= \partial_\alpha(T_0^n) \circ T_1 + (T_0^n)' \circ T_1 \cdot \partial_\alpha T_1 \\
(T_0^n \circ T_1)'' &= (T_0^n)'' \circ T_1 \cdot (T_1')^2 + (T_0^n)' \circ T_1 \cdot T_1'' \\
\partial_\alpha(T_0^n \circ T_1)' &= \partial_\alpha(T_0^n)' \circ T_1 \cdot T_1' + (T_0^n)'' \circ T_1 \cdot T_1' \cdot \partial_\alpha T_1 + (T_0^n)' \circ T_1 \cdot \partial_\alpha T_1'
\end{aligned}$$

which we may write as a matrix

$$\begin{pmatrix} T'_\omega \\ \partial_\alpha T_\omega \\ T''_\omega \\ \partial_\alpha T'_\omega \end{pmatrix} = \begin{pmatrix} T_1' & 0 & 0 & 0 \\ \partial_\alpha T_1 & 1 & 0 & 0 \\ T_1'' & 0 & (T_1')^2 & 0 \\ \partial_\alpha T_1' & 0 & T_1'\partial_\alpha T_1 & T_1' \end{pmatrix} \cdot \begin{pmatrix} (T_0^n)' \circ T_1 \\ \partial_\alpha(T_0^n) \circ T_1 \\ (T_0^n)'' \circ T_1 \\ \partial_\alpha(T_0^n)' \circ T_1 \end{pmatrix}.$$

By the same logic we may write

$$\begin{pmatrix} T'_\omega \\ \partial_\alpha T_\omega \\ T''_\omega \\ \partial_\alpha T'_\omega \end{pmatrix} = \begin{pmatrix} T_1' & 0 & 0 & 0 \\ \partial_\alpha T_1 & 1 & 0 & 0 \\ T_1'' & 0 & (T_1')^2 & 0 \\ \partial_\alpha T_1' & 0 & T_1'\partial_\alpha T_1 & T_1' \end{pmatrix}$$
$$\cdot \begin{pmatrix} T_0' \circ T_1 & 0 & 0 & 0 \\ \partial_\alpha T_0 \circ T_1 & 1 & 0 & 0 \\ T_0'' \circ T_1 & 0 & (T_0' \circ T_1)^2 & 0 \\ \partial_\alpha T_0' \circ T_1 & 0 & T_0' \circ T_1 \partial_\alpha T_0 \circ T_1 & T_0' \circ T_1 \end{pmatrix} \begin{pmatrix} (T_0^{n-1})' \circ T_0 \circ T_1 \\ \partial_\alpha(T_0^{n-1}) \circ T_0 \circ T_1 \\ (T_0^{n-1})'' \circ T_0 \circ T_1 \\ \partial_\alpha(T_0^{n-1})' \circ T_0 \circ T_1 \end{pmatrix}.$$

and use induction to give a series of matrices such that

$$\begin{pmatrix} T'_\omega \\ \partial_\alpha T_\omega \\ T''_\omega \\ \partial_\alpha T'_\omega \end{pmatrix} = \begin{pmatrix} T'_1 & 0 & 0 & 0 \\ \partial_\alpha T_1 & 1 & 0 & 0 \\ T''_1 & 0 & (T'_1)^2 & 0 \\ \partial_\alpha T'_1 & 0 & T'_1 \partial_\alpha T_1 & T'_1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} T'_0 \circ T_1 & 0 & 0 & 0 \\ \partial_\alpha T_0 \circ T_1 & 1 & 0 & 0 \\ T''_0 \circ T_1 & 0 & (T'_0 \circ T_1)^2 & 0 \\ \partial_\alpha T'_0 \circ T_1 & 0 & T'_0 \circ T_1 \partial_\alpha T_0 \circ T_1 & T'_0 \circ T_1 \end{pmatrix} \cdots \begin{pmatrix} T'_0 \circ T_0^{n-1} \circ T_1 \\ \partial_\alpha T_0 \circ T_0^{n-1} \circ T_1 \\ T''_0 \circ T_0^{n-1} \circ T_1 \\ \partial_\alpha T'_0 \circ T_0^{n-1} \circ T_1 \end{pmatrix}.$$

Using the explicit formulas for our branches we have

$$\begin{aligned} T_0 &= x(1 + (2x)^\alpha) \\ T_1 &= 2x - 1 \\ T'_0 &= 1 + (1+\alpha)(2x)^\alpha \\ T'_1 &= 2 \\ \partial_\alpha T_0 &= (\log(x) + \log(2))2^\alpha x^{\alpha+1} \\ \partial_\alpha T_1 &= 0 \\ T''_0 &= \alpha(1+\alpha)2^\alpha x^{\alpha-1} \\ T''_1 &= 0 \\ \partial_\alpha T'_0 &= (2x)^\alpha((\alpha+1)(\log(x) + \log(2)) + 1) \\ \partial_\alpha T'_1 &= 0 \end{aligned}$$

and we are able to calculate explicitly the values A_0, B_0, a_ω and b_ω. To calculate a_ω and b_ω we use $g_\omega = g_1 \circ g_0^{n-1}$ and we use $T_0^m \circ T_1 \circ g_1 \circ g_0^{n-1} = g_0^{n-1-m}$ to calculate

$$\begin{pmatrix} T'_\omega \circ g_\omega \\ \partial_\alpha T_\omega \circ g_\omega \\ T''_\omega \circ g_\omega \\ \partial_\alpha T'_\omega \circ g_\omega \end{pmatrix} = \begin{pmatrix} T'_1 \circ g_\omega & 0 & 0 & 0 \\ \partial_\alpha T_1 \circ g_\omega & 1 & 0 & 0 \\ T''_1 \circ g_\omega & 0 & (T'_1)^2 \circ g_\omega & 0 \\ \partial_\alpha T'_1 \circ g_\omega & 0 & T'_1 \circ g_\omega \partial_\alpha T_1 \circ g_\omega & T'_1 \circ g_\omega \end{pmatrix}$$

$$\cdot \begin{pmatrix} T'_0 \circ g_0^n & 0 & 0 & 0 \\ \partial_\alpha T_0 \circ g_0^n & 1 & 0 & 0 \\ T''_0 \circ g_0^n & 0 & (T'_0 \circ g_0^n)^2 & 0 \\ \partial_\alpha T'_0 \circ g_0^n & 0 & T'_0 \circ g_0^n \partial_\alpha T_0 \circ g_0^n & T'_0 \circ g_0^n \end{pmatrix} \cdots \begin{pmatrix} T'_0 \circ g_0 \\ \partial_\alpha T_0 \circ g_0 \\ T''_0 \circ g_0 \\ \partial_\alpha T'_0 \circ g_0 \end{pmatrix}$$

where we calculate g_0^m using the shooting method from section 7.1.1.

References

1. Antown, F., Dragičević, D., Froyland, G.: Optimal linear responses for Markov chains and stochastically perturbed dynamical systems. J. Stat. Phys. **170**(6), 1051–1087 (2018)
2. Antown, F., Froyland, G., Junge, O.: Linear response for the dynamic Laplacian and finite-time coherent sets. Nonlinearity **34**, 3337 (2019)
3. Aspenberg, M., Baladi, V., Leppänen, J., Persson, T.: On the fractional susceptibility function of piecewise expanding maps. arXiv Preprint (2019). arXiV:1910.00369
4. Bahsoun, W., Saussol, B.: Linear response in the intermittent family: differentiation in a weighted C^0-norm. Discrete Contin. Dynam. Syst. **36**(12), 6657–6668 (2016)

5. Bahsoun, W., Bose, C., Duan, Y.: Rigorous pointwise approximations for invariant densities of non-uniformly expanding maps. Ergod. Theory Dyn. Syst. **35**, 1028–1044 (2013). https://doi.org/10.1017/etds.2013.91
6. Bahsoun, W., Galatolo, S., Nisoli, I., Niu, X.: A rigorous computational approach to linear response. Nonlinearity **31**(3), 1073–1109 (2018)
7. Bahsoun, W., Ruziboev, M., Saussol, B.: Linear response for random dynamical systems. Adv. Math. **364**, 107011 (2020)
8. Baladi, V.: On the susceptibility function of piecewise expanding interval maps. Commun. Math. Phys. **275**, 839–859 (2007)
9. Baladi, V.: Linear response, or else. In: Proceedings of the International Congress of Mathematicians, Seoul 2014. vol. III, pp. 525–545. Kyung Moon Sa, Seoul (2014)
10. Baladi, V., Smania, D.: Fractional susceptibility functions for the quadratic family: Misiurewicz–Thurston parameters. Commun. Math. Phys. **385**, 1957–2007 (2020)
11. Baladi, V., Todd, M.: Linear response for intermittent maps. Commun. Math. Phys. **347**(3), 857–874 (2016)
12. Butterley, O., Liverani, C.: Smooth Anosov flows: correlation spectra and stability. J. Mod. Dyn. **1**, 301–322 (2007)
13. Butterly, O., Kiamari, N., Liverani, C.: Locating Ruelle–Pollicott resonances. arXiv Preprint (2012). arXiv:2012.13145
14. Chandramoorthy N., Wang Q.: A computable realization of Ruelle's formula for linear response of statistics in chaotic systems. arXiv Preprint (2002). arXiv:2002.04117
15. Chandramoorthy N., Wang Q.: Efficient computation of linear response of chaotic attractors with one-dimensional unstable manifolds. arXiv Preprint (2013). arXiv:2103.08816
16. Choudhury, B.: The Riemann zeta-function and its derivatives. Proc. Math. Phys. Sci. **450**, 477–499 (1995)
17. de Lima, A., Smania, D.: Central limit theorem for the modulus of continuity of averages of observables on transversal families of piecewise expanding unimodal maps. J. Inst. Math. Jussieu **17**(3), 673–733 (2018)
18. Dolgopyat, D.: On differentiability of SRB states for partially hyperbolic systems. Invent. Math. **155**, 389–449 (2004)
19. Dragičević, D., Sedro, J.: Statistical stability and linear response for random hyperbolic dynamics. Ergod. Theory Dyn. Syst. **43**(2), 515–544 (2020)
20. Frigo, M., Johnson, S.G.: The design and implementation of FFTW3. Proc. IEEE **93**(2), 216–231 (2005)
21. Galatolo, S., Giulietti, P.: A linear response for dynamical systems with additive noise. Nonlinearity **32**(6), 2269–2301 (2019)
22. Galatolo, S., Nisoli, I.: An elementary approach to rigorous approximation of invariant measures. SIAM J. Appl. Dyn. Syst. **13**(2), 958–985 (2014)
23. Galatolo, S., Pollicott, M.: Controlling the statistical properties of expanding maps. Nonlinearity **30**(7), 2737–2751 (2017)
24. Galatolo, S., Sedro, J.: Quadratic response of random and deterministic dynamical systems. Chaos **30**, 023113 (2020)
25. Galatolo, S., Nisoli, I., Saussol, S.: An elementary way to rigorously estimate convergence to equilibrium and escape rates. J. Comput. Dyn. **2**(1), 51–64 (2015)
26. Galatolo, S., Monge, M., Nisoli, I., Poloni, F.: A general framework for the rigorous computation of invariant densities and the course-fine strategy. Chaos Solitons Fractals **170**, 113329 (2023)
27. Gottwald, G.: Introduction to focus issue: linear response theory: potentials and limits. Chaos **30**(2), 020401 (2020). https://doi.org/10.1063/5.0003135
28. Gouëzel, S., Liverani, C.: Banach spaces adapted to Anosov systems. Ergod. Theory Dyn. Syst. **26**, 189–217 (2006)
29. Gutiérrez, M.S., Lucarini, V.: Response and sensitivity using Markov chains. J. Stat. Phys. **179**(5–6), 1572–1593 (2020)
30. Heideman, M., Johnson, D., Burrus, C.: Gauss and the history of the fast Fourier transform. IEEE ASSP Mag. **1**(4), 14–21 (1984)
31. Higham, N.J.: Accuracy and Stability of Numerical Algorithms, 2nd edn. SIAM, Philadelphia (2002)
32. Jézéquel, M.: Parameter regularity of dynamical determinants of expanding maps of the circle and an application to linear response. Discrete Contin. Dyn. Syst. **39**(2), 927–958 (2019)
33. Katok, A., Knieper, G., Pollicott, M., Weiss, H.: Differentiability and analyticity of topological entropy for Anosov and geodesic flows. Invent. Math. **98**(3), 581–597 (1989)
34. Kloeckner, B.: The linear request problem. Proc. Am. Math. Soc. **146**(7), 2953–2962 (2018)

35. Koltai, P., Lie, H.-C., Plonka, M.: Fréchet differentiable drift dependence of Perron–Frobenius and Koopman operators for non-deterministic dynamics. Nonlinearity **32**(11), 4232–4257 (2019)
36. Korepanov, A.: Linear response for intermittent maps with summable and nonsummable decay of correlations. Nonlinearity **29**(6), 1735–1754 (2016)
37. Ledoux, V., Moroz, G.: Evaluation of Chebyshev polynomials on intervals and application to root finding. In: Mathematical Aspects of Computer and Information Sciences 2019, November 2019, Gebze, Turkey. hal-02405752 (2019)
38. Liverani, C., Saussol, B., Vaienti, S.: A probabilistic approach to intermittency. Ergod. Theory Dyn. Syst. **19**, 671–685 (1999)
39. Lucarini, V.: Response operators for Markov processes in a finite state space: radius of convergence and link to the response theory for Axiom A systems. J. Stat. Phys. **162**(2), 312–333 (2016)
40. Ni, A.: Linear response algorithm for differentiating stationary measures of chaos. arXiv preprint (2020). arXiv:2009.00595
41. Ni, A.: Approximating linear response by nonintrusive shadowing algorithms. SIAM J. Numer. Anal. **59**(6), 2843–2865 (2021)
42. Pollicott, M., Vytnova, P.: Linear response and periodic points. Nonlinearity **29**(10), 3047–3066 (2016)
43. Ruelle, D.: Differentiation of SRB states. Commun. Math. Phys. **187**, 227–241 (1997)
44. Sedro, J.: Pre-threshold fractional susceptibility functions at Misiurewicz parameters. Nonlinearity **34**(10), 7174 (2020)
45. Sélley, F., Tanzi, M.: Linear response for a family of self-consistent transfer operators. Commun. Math. Phys. **382**(3), 1601–1624 (2020)
46. Trefethen, L.N.: Approximation Theory and Approximation Practice. SIAM, Philadelphia (2018)
47. Tucker, W.: Validated Numerics: A Short Introduction to Rigorous Computations. Princeton University Press, Princeton (2011)
48. Wormell, C.L.: Spectral Galerkin methods for transfer operators in uniformly expanding dynamics. Numer. Math. **14**, 421–463 (2019)
49. Wormell, C.L.: Efficient computation of statistical properties of intermittent dynamics. arXiv Preprint (2021). arXiv:2106.01498
50. Wormell, C., Gottwald, G.: Linear response for macroscopic observables in high-dimensional systems. Chaos **29**, 113–127 (2019)
51. Xiang, S., Chen, X., Wang, H.: Error bounds for approximation in Chebyshev points. Numer. Math. **116**, 463–491 (2010). https://doi.org/10.1007/s00211-010-0309-4