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Rigorous approximation of stationary measures and convergence to equilibrium for iterated function systems*

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Abstract

We study the problem of the rigorous computation of the stationary measure and of the rate of convergence to equilibrium of an iterated function system described by a stochastic mixture of two or more dynamical systems that are either all uniformly expanding on the interval, either all contracting. In the expanding case, the associated transfer operators satisfy a Lasota–Yorke inequality, we show how to compute a rigorous approximations of the stationary measure in the L^1 norm and an estimate for the rate of convergence. The rigorous computation requires a computer-aided proof of the contraction of the transfer operators for the maps, and we show that this property propagates to the transfer operators of the IFS. In the contracting case we perform a rigorous approximation of the stationary measure in the Wasserstein–Kantorovich distance and rate of convergence, using the same functional analytic approach. We show that a finite computation can produce a realistic computation of all contraction rates for the whole parameter space. We conclude with a description of the implementation and numerical experiments.

Keywords: rigorous numerical methods, stationary measure, iterated function system, convergence to equilibrium
Mathematics Subject Classification: 37M25, 37H

(Some figures may appear in colour only in the online journal)

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1. Introduction

The reliable simulation and forecasting of the statistical properties of a chaotic dynamical model is a difficult and important task. Our investigation is focused towards the possibility of analyzing a dynamical system in an automated way, forecasting long time statistical features and, more in general, any rigorous information that can be useful for computer-aided proofs. Here we consider the rigorous approximation of the stationary measure of random dynamical systems and its speed of convergence to equilibrium. By rigorous approximation we mean that the result of the computation is mathematically certified, up to an explicitly given error.

By *iterated function system* (IFS) we mean the datum of a space X and a finite collection of transformations $T_i : X \to X$ for $i \in I$, plus a set of positive parameters (probabilities) p_i for $i \in I$, summing up to 1. Iteratively, one of the maps T_i is applied on the set X, chosen with probability p_i independently of the previous steps.

In this article we study the problem of computing effectively, with a rigorous bound on the error, the stationary (invariant) measure of such an IFS and its rate of convergence to equilibrium. In the literature the problem of computation of the stationary measure for such systems was considered (see e.g. [Bla01, Fro99, Obe05, Swi13]) proving the convergence of the approximating algorithms to the correct estimation, and sometime giving some more information on the behavior of the approximation error. In particular we remark that in [Obe05] asymptotic estimations on the behavior of the approximation error are given whitout an explicit estimation for all the constants involved, while in [Fro99] a way to get explicits bounds on the approximation error for the stationary measure is given in the case of random iteration of expanding maps of the circle. The paper [Bla01] consider the approximation of the spectrum, which is a related concept, but to our knowledge no rigorous estimates for the rate of convergence to equilibrium, as we describe in section 4, have been implemented in this context before. We show how, with the help of some general tools that have been developed in [GN14, GS15] for the study of transfer operators, we can implement an effective strategy to perform these tasks on random systems and we apply it to nontrivial examples, presenting the result of some rigorous computation, which gives rigorous information on the behavior of the example being considered.

We will use a functional analytic approach, assuming the phase space X to be a metric space equipped with the Borel σ-algebra (which we assume to be preserved by the inverse of the T_i maps), and considering the transfer operator acting on Borel signed measures. For a deterministic transformation T the corresponding transfer operator L_T acting on probability measures is defined as

$$(L_T \mu)(A) = \mu(T^{-1}(A)),$$

and when μ represents a probability distribution in X, then $L_T \mu$ represents the probability distribution on X after one application of the transformation T.

Let us consider an IFS constructed with the maps T_i having transfer operators L_i and probabilities p_i. The (annealed) transfer operator associated to such an IFS is defined by

$$L = \sum_{i \in I} p_i L_i. \tag{1}$$

A fixed point for this transfer operator is called *stationary measure* for the IFS (see e.g. [KL06, Pel84]). In this paper we decribe a way to find this fixed point and other features of the underlying system by the properties of the operator L. Part of the information on L that will be used comes from the qualitative properties of maps T_i, the probabilities p_i and part will be computed in a rigorous way by a suitable finite rank approximation of L.

Let us introduce some notation: on a suitable space of measures $\mathcal{B}$, we will denote by $V_{\mathcal{B}}$ (or just V when no confusion is possible) the set of *zero-average measures*, that is

$$V_{\mathcal{B}} = \left\{ \mu \in \mathcal{B} : \int_X 1 \mathrm{d}\mu = 0 \right\}.$$

When working on the interval, we denote by $\|\cdot\|_{BV}$ the norm on measures defined as

$$\|\mu\|_{BV} = \sup_{\phi \in C^1 : \|\phi\|_\infty = 1} \int_X \phi' \mathrm{d}\mu,$$

the measures having finite norm are absolutely continuous with bounded variation density (see [Liv04]).

We will abuse notation by extending the L^1 norm to signed measures as

$$\|\mu\|_{L^1} = \mu^+(X) + \mu^-(X) \tag{2}$$

for a Hahn decomposition $\mu = \mu^+ - \mu^-$ [Hal13], it coincides with the L^1 norm of the density when μ is absolutely continuous.

For simplicity we will consider two main cases: the case where the maps T_i are piecewise expanding on the interval, and the case where T_i are contracting. More general cases can be considered where the system is averagely expanding or contracting, (see [Bla01, Pel84]) and the theory described here adapts to many of these cases, since our approach is based directly on the properties of the transfer operator. The technique we use is based on the fact that both in these two expanding and contracting cases, the operator L is a contraction in a suitable sense on a suitable space.

- The first case we consider is the case of uniformly expanding transformations on the interval; in such a case for a transformation T that is piecewise C^2 we have a Lasota–Yorke inequality with respect to the BV and L^1 norms

  $$\|L\mu\|_{BV} \leqslant \lambda \|\mu\|_{BV} + B\|\mu\|_{L^1} \tag{3}$$

  for constants $\lambda < 1$ and B, which will be explicitly estimated. If the system has a unique absolutely continuous invariant measure, then the iterates of L eventually contract V_{BV}. It is possible to find a suitable finite rank approximation of L satisfying the same property (equation (3)) and having a unique fixed point which is close to the unique invariant measure of L.
- The second case is the classic case of contracting maps, on a generic bounded subset of $\mathbb{R}^n$. In such a case the transfer operator is a contraction in the dual space of Lipschitz function, that is, the space of measures with respect to the norm induced by Wasserstein–Kantorovich distance.

All software developed in this work is publicly available as free software, as detailed in section 7.

2. Summary

We present here an outline of the results.

In section 3 we work out the abstract strategy for the approximation of the stationary measure, that can be used in both the expanding and contracting case.

In section 4 we show the general strategy for the estimation of the rate of convergence to equilibrium for operators satisfying a Lasota–Yorke inequality.

In section 5 we prove that contraction properties are preserved for nearby operators, and that proving contraction on the operators L_i is indeed enough to get a contraction of the IFS for any choice of the probabilities. In practice the estimation of the contraction rate obtained via the *a priori* formula can be quite pessimistic, but still, it can be used to have a usable estimate, as it will be shown in an example.

In sections 6 and 8 we demonstrate the application of the above results in the case of the L^1 approximation of the invariant measure and contraction rate for IFS formed by uniformly expanding transformations of the interval.

In section 9 we treat the more classical case of contracting transformations, approximating the invariant measure in the Wasserstein–Kantorovich metric.

3. Strategy for rigorous approximation of the stationary measure

A stationary measure measure of the system is a fixed point of the transfer operator defined in (1). To approximate it with a certified bound on the error we will use a quantitative fixed point stability result used in [GN14] for the approximation of the invariant measure of deterministic dynamical systems.

Let L be a transfer operator acting on a Banach space $\mathcal{B}$ of Borel measures and having fixed point μ. We will use a finite-rank projection $\pi_\delta : \mathcal{B} \to \mathcal{B}$ (describing a finite dimensional approximation of measures), and denote by L_δ the approximated operator $\pi_\delta L \pi_\delta$, and suppose μ_δ is its unique fixed point. Note that $\pi_\delta L \pi_\delta$ is a stochastic matrix, so the biggest eigenvalue is 1 and the corresponding eigenvector is a fixed point. We assume that L and L_δ preserve the space of zero-average measures $V_\mathcal{B}$.

While performing a computation, L_δ is computable with rigorous error and representable in a suitable basis as a matrix of floating point numbers on computer, and μ_δ can be rigorously approximated as a vector.

The following theorem can be used to compute rigorously the error of approximating μ by μ_δ with respect to the norm $\|\cdot\|_\mathcal{B}$, it is essentially theorem 1 of [GN14].

Theorem 3.1. *Suppose that*

(1) $\|(L_\delta - L)\mu\|_\mathcal{B} < \infty$,
(2) $\exists N$ *such that* $\|L_\delta^N\|_{V_\mathcal{B} \to V_\mathcal{B}} \leqslant \alpha < 1$,
(3) $\|L_\delta^i\|_{\mathcal{B}\to\mathcal{B}} \leqslant C_i$ *for* $i = 0, \ldots, N-1$.
Then

$$\|\mu_\delta - \mu\|_\mathcal{B} \leqslant \|(L_\delta - L)\mu\|_\mathcal{B} \cdot \frac{1}{1-\alpha} \sum_{i=0}^{N-1} C_i$$

Proof. See [GN14]. □

It easily follows

Corollary 3.2. *If L_δ is a contraction and $\|L_\delta\|_{\mathcal{B}} \leqslant \gamma < 1$, then*

$$\|\mu_\delta - \mu\|_{\mathcal{B}} \leqslant \frac{1}{1-\gamma}\|(L_\delta - L)\mu\|_{\mathcal{B}}.$$

Remark 3.3. Both theorem 3.1 and corollary 3.2 are still valid when $\|\cdot\|_{\mathcal{B}}$ is only defined on the space of measures of zero average. This is very useful for working with Wasserstein–Kantorovich distance, which only defines a norm on measures with zero average.

We can see, the invariant measure that is being approximated appears in theorem 3.1. Indeed, item 1 in the theorem requires an estimation that depends on the *a priori* properties of the invariant density μ and on the precision of the approximation of L by L_δ.

In practice, this can be done using a stronger norm $\mathcal{B}'$ that can be used to upper bound the norm of the invariant measure μ, and such that $\|L_\delta - L\|_{\mathcal{B}'\to\mathcal{B}}$ can be made arbitrarily small, so that

$$\|L_\delta\mu - L\mu\|_{\mathcal{B}} \leqslant \|L_\delta - L\|_{\mathcal{B}'\to\mathcal{B}} \cdot \|\mu\|_{\mathcal{B}'}.$$

Such hypotheses are available when a Lasota–Yorke inequality involving $\mathcal{B}'$ and $\mathcal{B}$ is satisfied.

Informally, the algorithm we are going to implement for the approximation of the stationary measures can be described as follows:

- Input the maps $\{T_i\}$, the probabilities $\{p_i\}$, and the partition.
- For each map T_i compute the matrix P_i approximating $L_{i,\delta}$.
- Compute L_δ as linear combination of the $\{L_{i,\delta}\}$ with the $\{p_i\}$ as coefficients.
- Compute the approximated fixed point μ_δ of L_δ up to some required approximation ϵ_1.
- Compute an estimation for $\|L_\delta\mu - L\mu\|_{\mathcal{B}}$ up to some error ϵ_2, as required by 1 of theorem 3.1.
- Compute N such that item 2 of theorem 3.1 is verified, in practice we compute the smallest N such that $\|(P_i|_V)^N\|_{\mathcal{B}\to\mathcal{B}} \leqslant \frac{1}{2}$ (alternatively estimate such N via theorem 5.2).
- Estimate the C_i of 3 of theorem 3.1 (for $\mathcal{B} = L^1$ they are all $\leqslant 1$, so this step can be skipped).
- If all computations end successfully, output μ_δ and the error rigorously estimated via theorem 3.1.

Items 2 and 3 of theorem 3.1 can be verified computationally, but their estimation is neither trivial nor rapid, so we developed a strategy that permits to give *a priori* bound for these items for any combination of the $\{p_i\}$ once we have estimated items 2 and 3 for some specific choices of the parameters; section 5 explains this strategy.

We will illustrate the algorithm in particular cases in sections 8 and 10.

4. Recursive convergence to equilibrium estimation for maps satisfying a Lasota–Yorke inequality

Beyond a rigorous approximation of a stationary measures of some example of IFS, it is important to check if this stationary measure is unique, and obtain an estimate for the rate of convergence to this measure. For the contracting IFS the uniqueness is well known (see [Bar98] e.g.) and the rate of convergence is upper estimated by the smallest contraction rate of

the maps involved (see section 10). For the piecewise expanding ones we apply a technique [GS15], which works in general for systems safisfying a Lasota–Yorke inequality.

We will see how, the Lasota–Yorke inequality together with a suitable approximation of the system by a finite dimensional one can be used to deduce finite time and asymptotic upper bounds on the contraction of the zero average space. This method will be then applied to expanding IFS, giving as a result the uniqueness of the absolutely continuous invariant measure in the space defined by the strong norm appearing in the Lasota–Yorke inequality (as all the other measure will converge to it) and an effective quantitative estimation for the rate of convergence.

Consider two vector subspaces of the space of signed measures $B_s \subseteq B_w$ with norms $\| \|_s \geqslant \| \|_w$.

Definition 4.1. Let $\phi(n)$ be a real sequence converging to zero. We say that the system shows *convergence to equilibrium* with respect to norms $\| \|_w, \| \|_s$ and speed ϕ if $\forall\, g \in V_s$ (the space of zero-average functions with respect to $\| \|_s$, as defined in the introduction)

$$\|L_0^n(g)\|_w \leqslant \phi(n)\|g\|_s. \tag{4}$$

Let us suppose that there are operators L_δ approximating L satisfying an approximation inequality of the following type: there are constants C, D such that $\forall\, g \in B_s$, $\forall\, n \geqslant 0$:

$$\|(L_\delta^n - L^n)g\|_w \leqslant \delta(C\|g\|_s + nD\|g\|_w). \tag{5}$$

We note that if we consider piecewise expanding maps and the Ulam discretization this inequality can be proved, and the coefficient explicitly estimated (see [GS15]). Now let us consider as before the space $V_{\mathcal{B}}$ of zero total mass measures and let us suppose that there exists δ and n_1 such that

$$\forall\, v \in V_{\mathcal{B}},\ \ \|L_\delta^{n_1}(v)\|_w \leqslant \lambda_2\|v\|_w \tag{6}$$

with $\lambda_2 < 1$. Let us consider a starting measure: $g_0 \in V$, let us denote $g_{i+1} = L^{n_1}g_i$. If the system is as above, putting together the Lasota–Yorke inequality, (5) and (6)

$$\begin{cases} \|L^{n_1}g_i\|_s \leqslant A\lambda_1^{n_1}\|g_i\|_s + B\|g_i\|_w, \\ \|L^{n_1}g_i\|_w \leqslant \lambda_2\|g_i\|_w + \delta(C\|g_i\|_s + n_1D\|g_i\|_w). \end{cases}$$

Compacting it in a vector notation

$$\begin{pmatrix} \|g_{i+1}\|_s \\ \|g_{i+1}\|_w \end{pmatrix} \preceq \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix} \begin{pmatrix} \|g_i\|_s \\ \|g_i\|_w \end{pmatrix}, \tag{7}$$

where $\preceq$ indicates the component-wise $\leqslant$ relation (both coordinates are less or equal). The relation $\preceq$ can be used because the matrix is positive. The relation (7) and the assumptions allow to explicitly estimate the contraction rate, by approximating the matrix and its iterations. Let $\mathcal{M} = \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix}$. Consequently, we can bound $\|g_i\|_s$ and $\|g_i\|_w$ by a sequence

$$\begin{pmatrix} \|g_i\|_s \\ \|g_i\|_w \end{pmatrix} \preceq \mathcal{M}^i \begin{pmatrix} \|g_0\|_s \\ \|g_0\|_w \end{pmatrix} \tag{8}$$

which can be computed explicitly. This gives an explicit estimate on the speed of convergence for the norms $\| \|_s$ and $\| \|_w$ at a given time.

Remark 4.2. We remark that, $\lambda_1^{n_1}$, $\lambda_2 < 1$ and the quantities δC, $\delta n_1 D$ have a chance to be very small when δ is very small. This is not trivial because n_1 depend on δ. However, in the case of piecewise expanding maps, with L_δ being an Ulam-type approximation of L, as we consider in this paper, $\delta n_1 D$ can be made sufficiently small (see [GS15], theorem 12).

A rigorous explicit estimation for the asymptotic behavior of (8) comes from eigenvalues and eigenvectors of $\mathcal{M}$. Indeed, let the leading eigenvalue be denoted by $\rho_{\mathcal{M}}$ and a left positive eigenvector (a, b), such that $a + b = 1$. For each pair of values (a, b) such that $a + b = 1$ we can define a norm

$$\|g\|_{(a,b)} = a\|g\|_s + b\|g\|_w .$$

We have that

$$\|Lg\|_{(a,b)} = a\|Lg\|_s + b\|Lg\|_w \leqslant (a, b) \cdot \mathcal{M} \cdot \begin{pmatrix} \|g\|_s \\ \|g\|_w \end{pmatrix}$$

then

$$\|L^{kn_1} g\|_{(a,b)} \leqslant \rho_{\mathcal{M}}^k \|g\|_{(a,b)} .$$

We do not expect the numbers and vectors computed in this way to be optimal, but they give explicit upper estimates for the convergence to equilibrium of the system (see section 8.1).

5. Uniform decay time estimations for the discretized operators

When considering discretized operators to apply theorem 3.1 a particularly time consuming task is to compute its decay rates needed in item 2. In this section we show how this checking can be done once and for all at the level of the maps T_i, obtaining an estimation which is valid for every choice of the values of the parameters $\{p_i\}$ in the definition of the transfer operator we consider (see equation (1)).

For this we take advantage of the contraction property of the (discretized) operator on the zero-average subspace being preserved in a neighborhood. We will consider a general norm $\|\cdot\|$, that we will take to be L^1 in the expanding case, and the norm induced by the Wasserstein–Kantorovich distance in the contracting case.

In this section we assume L to be the discretized transfer operator restricted to the space of zero-average functions. Assume $\tilde{L}$ be a nearby (discretized) operator, in the sense that the operator norm $\|L - \tilde{L}\|$ is small. Let n be such that $\|L^n\| < 1/4$. Then

Proposition 5.1. *Assume* $\|L^n\| < \frac{1}{4}$, *and that* $\|L^i\| < C$ *for each* i. *If* $\tilde{L}$ *is another operator such that* $\|\tilde{L} - L\| < \frac{1}{n4C^2}$, *and such that* $\|\tilde{L}^i\| < C$ *for each* i, *then* $\|\tilde{L}^n\| \leqslant \frac{1}{2}$.

Proof. Indeed:

$$\begin{aligned}\|\tilde{L}^n\| &= \|L^n + (\tilde{L}^n - L^n)\| \\ &\leqslant \|L^n\| + \|\tilde{L}^n - L^n\| \\ &\leqslant \|L^n\| + \sum_{i=0}^{n-1} \|\tilde{L}^i(\tilde{L} - L)L^{n-i-1}\| \\ &\leqslant \|L^n\| + \sum_{i=0}^{n-1} \|\tilde{L}^i\| \cdot \|\tilde{L} - L\| \cdot \|L^{n-i-1}\| \\ &\leqslant \|L^n\| + nC^2 \cdot \|\tilde{L} - L\| \\ &\leqslant \frac{1}{2}.\end{aligned}$$

□

The above proposition ensures that for each contracting operator, all nearby operators are also contractions. The space describing all possibile probabilities is compact, and consequently assuming all combinations of the operators L_i to be contractions (taking a power if necessary), we could use compactness to prove it in a finite number of steps for a sufficiently fine grid of possibilities, each step granting the contraction in a neighborhood.

However, we have that in the IFS case all operators are convex combinations of contractions (in a certain number of steps), and this information can be used at once to bound the contraction time of a combination. We start working with the combination of two operators. We remark that a combination of any finite number of operator can be seen as obtained taking inductively a convex combination of two operators that are contractions, so it is possible to use the theorem that follows to work with any finite number of operators.

Assume L_0 and L_1 to be operators on a Banach space, and for a sequence $\omega = (\omega_1, \omega_2, ..., \omega_k)$ with $\omega_i \in \{0, 1\}$, denote $L^\omega = L_{\omega_1} L_{\omega_2} ... L_{\omega_k}$. We also denote by $|\omega|$ its length k, and by $|\omega|_0, |\omega|_1$ the number of occurrences of 0 or 1 in ω, respectively. Assume that

$$\forall\, \omega \|L^\omega\| < C, \qquad \|L_0^{n_0}\| < \frac{1}{2C}, \qquad \|L_1^{n_1}\| < \frac{1}{2C}.$$

for some $C > 0$ and n_0, n_1 big enough.

Theorem 5.2. *Let* $p \in [0, 1]$, *then*

$$\|(pL_0 + (1 - p)L_1)^M\| < \frac{1}{2}$$

for all M *satisfying the lower bound*

$$M \geqslant N - 1 + N\frac{\log 2C}{-\log\left(1 - \frac{p^{n_0}}{2} - \frac{(1-p)^{n_1}}{2}\right)}.$$

for $N = \max\{n_0, n_1\}$.

Proof. Let us expand the Mth power in all possible compositions of L_0 and L_1, indexed by all words ω of length M:

$$L^M = (pL_0 + (1-p)L_1)^M = \sum_{\omega:|\omega|=M} p^{|\omega|_0}(1-p)^{|\omega|_1}L^\omega.$$

Estimating the norm of L^ω for a given ω, we can start with the uniform estimation C, and for each occurrence of 0^{n_0} or 1^{n_1} in ω we can account a contribution of an extra factor $\frac{1}{2C}\cdot C = \frac{1}{2}$ to the estimate. That is, $\|L^\omega\| \leqslant C\cdot 2^{-H(\omega)}$, where $H(\omega)$ denotes the number of occurrences of either 0^{n_0} or 1^{n_1} in ω. Consequently, the norm in the claim can be estimated with

$$S(M) = C\cdot \sum_{\omega:|\omega|=M} p^{|\omega|_0}(1-p)^{|\omega|_1}2^{-H(\omega)}.$$

To estimate $S(M)$, we will proceed by induction on M and denote by $S_0(M)$ and $S_1(M)$ the same sum restricted to the ω satisfying $\omega_1 = 0$ or 1 respectively. Decomposing the sum depending on the biggest number of initial 0's or 1's in ω, we have

$$\begin{aligned} S(M) \leqslant & \sum_{i=1}^{n_0-1} p^i S_1(M-i) + \frac{1}{2}p^{n_0}S(M-n_0) \\ & + \sum_{i=1}^{n_1-1}(1-p)^i S_0(M-i) + \frac{1}{2}(1-p)^{n_1}S(M-n_1). \end{aligned}$$

Considering that for each $i > 1$

$$S_0(i) < pS(i-1), \qquad S_1(i) < (1-p)S(i-1),$$

we can estimate

$$\begin{aligned} S(M) \leqslant & \sum_{i=1}^{n_0-1} p^i(1-p)S(M-i-1) + \frac{1}{2}p^{n_0}S(M-n_0) \\ & + \sum_{i=1}^{n_1-1} p(1-p)^i S(M-i-1) + \frac{1}{2}(1-p)^{n_1}S(M-n_1). \end{aligned}$$

We have now a sequence $S(M)$ which satisfies a recurrence inequality. It is natural to compare it with the sequence satisfying the exact recurrence (with equality), which will provide an upper bound on $S(M)$.

A recurrence where each next element is defined as a positive combination of previous terms can be estimated using the powers of the unique positive real root of the characteristic polynomial. This technique is standard in the theory of linear recurrence sequences (see for instance [Wil13]), and based on the following very simple idea: if the powers of the real number α satisfy a linear equation

$$\alpha^n = \sum_{i=0}^{n-1} c_i\alpha^i$$

with positive coefficients c_i (such α annihilates the polynomial, $X^n - c_{n-1}X^{n-1}...-c_0$, called characteristic polynomial of the recurrence), then whenever positive real numbers $x_0,\ldots,x_n$ satisfy

$$x_n = \sum_{i=0}^{n-1} c_i x_i$$

and satisfy $x_i \leqslant K\alpha^i$ for $0 \leqslant i < n$ and some constant K, then $x_n \leqslant K\alpha^n$. Iterating and using $K\alpha$, $K\alpha^2$, … in the role of K, if the x_i are defined by recurrence for $i > n$, that is for each $k > n$ we have

$$x_k = \sum_{i=0}^{n-1} c_i x_{k-n+i},$$

we obtain that $x_i \leqslant C\alpha^i$ for each $i \geqslant 0$.

Let $N = \max\{n_0, n_1\}$. In our case, the characteristic polynomial is

$$X^N = \sum_{i=1}^{n_0-1} p^i(1-p)X^{N-i} + \frac{1}{2}p^{n_0}X^{N-n_0} + \sum_{i=1}^{n_1-1} p(1-p)^i X^{N-i} + \frac{1}{2}(1-p)^{n_1}X^{N-n_1},$$

and such an equation has a real root which is <1 by intermediate value teorem, because the lhs is smaller than the rhs for $X = 0$, but becomes bigger for $X = 1$.

Such a root α should satisfy

$$\begin{aligned}\alpha^N &\leqslant \sum_{i=1}^{n_0-1} p^i(1-p) + \frac{1}{2}p^{n_0} + \sum_{i=1}^{n_1-1} p(1-p)^i + \frac{1}{2}(1-p)^{n_1} \\ &= \left[\sum_{i=1}^{n_0-1} p^i(1-p) + p^{n_0}\right] - \frac{1}{2}p^{n_0} + \left[\sum_{i=1}^{n_1-1} p(1-p)^i + (1-p)^{n_1}\right] - \frac{1}{2}(1-p)^{n_1} \\ &= 1 - \frac{1}{2}p^{n_0} - \frac{1}{2}(1-p)^{n_1},\end{aligned}$$

and observe that the sums under brackets are telescopic and can be simplified joining consecutive terms to obtain p and $1-p$ respectively. Consequently, we have

$$\alpha \leqslant \left(1 - \frac{1}{2}p^{n_0} - \frac{1}{2}(1-p)^{n_1}\right)^{1/N}.$$

Furthermore, we know that $S(i) \leqslant C < C\alpha^{i-N+1}$ for $0 \leqslant i < N$, and this implies that $S(M) < C\alpha^{M-N+1}$ for all M, applying the above reasoning. Consequently $S(M) < \frac{1}{2}$ whenever $\alpha^{M-N+1} \leqslant \frac{1}{2C}$, and it will be sufficient that

$$(M-N-1)\log(\alpha) \leqslant -\log(2C).$$

Dividing by $\log(\alpha)$ (which is negative) and taking into account the estimation we have for α we obtain the result whenever M satisfies the inequality

$$M \geqslant N - 1 + N\frac{\log 2C}{-\log\left(1 - \frac{p^{n_0}}{2} - \frac{(1-p)^{n_1}}{2}\right)}.$$

□

6. Estimation of the error in the uniformly expanding case

In this section we explain how to estimate item 1 in the uniformly expanding case. We recall what a Lasota–Yorke inequality is, and show that if such an inequality is satisfied by all the transformations T_i, then such an inequality can again be proved for a convex combination of their transfer operators.

Let X be the either unit interval $[0, 1]$, either S^1 (which we still identify to the unit interval with the additional identification $0 = 1$).

Definition 6.1. We say that a map T on the interval X is *piecewise expanding* if X can be partitioned in finite set of intervals where T is C^2 and $|T'| > 2$, and furthermore $\frac{T''}{(T')^2}$ is bounded.

We will work with transfer operators on measures satisfying a Lasota–Yorke inequality

$$\|L^N\mu\|_{\mathcal{B}'} \leqslant \lambda_1^N \|\mu\|_{\mathcal{B}'} + B_1\|\mu\|_{\mathcal{B}}.$$

Note that if all the transfer operators of the single maps in an IFS satisfy a Lasota–Yorke inequality, then the transfer operator of the IFS satisfies such an inequality too, as formalized in the following proposition.

Proposition 6.2. *Assume the operators L_i to satisfy the inequality*

$$\|L_i\mu\|_{\mathcal{B}'} \leqslant \lambda_i\|\mu\|_{\mathcal{B}'} + B_i\|\mu\|_{\mathcal{B}},$$

for $i = 1,\ldots,k$ then the convex combination $L = \sum_{i=1}^k p_i L_i$ satisfies

$$\|L\mu\|_{\mathcal{B}'} \leqslant \sum_{i=1}^k p_i\lambda_i\|\mu\|_{\mathcal{B}'} + \sum_{i=1}^k p_i B_i\|\mu\|_{\mathcal{B}}.$$

The proof is straightforward.

Remark 6.3. Please note that the fact that all the L_i satisfy such inequalities is a sufficient condition for the operator L to satisfy such an inequality: when all L_i satisfy such an inequality, then a convex combination also does. On the other hand, this condition is not necessary. When some L_i do not satisfy a Lasota–Yorke inequality, a combination may still satisfy such inequality for a suitable choice of the p_i. We remark that these this kind of 'averagely expanding' systems are already considered and studied in [Bla01, Pel84]. An interesting case could be with one operator being the transfer operator of an irrational rotation on the circle, and other operators satisfying a Lasota–Yorke inequality (see a concrete example in section 8).

Remark 6.4. Note also that if

$$\|L\mu\|_{\mathcal{B}'} \leqslant \lambda\|\mu\|_{\mathcal{B}'} + B\|\mu\|_{\mathcal{B}}.$$

then applying the inequality iteratively we have

$$\|L^N\mu\|_{\mathcal{B}'} \leqslant \lambda^N\|\mu\|_{\mathcal{B}'} + \frac{CB}{1-\lambda}\|\mu\|_{\mathcal{B}}.$$

for each N, where C is an upper bound for $\|L^i\|_{\mathcal{B}\to\mathcal{B}}$ for $0 < i < N$ (note that $C = 1$ for $\mathcal{B} = L^1$). Hence the Lasota–Yorke can always be recovered from the $N = 1$ case, up to replacing B with a bigger constant.

We cite here a few results on the existence of such inequality in the case of uniformly expanding maps of the interval.

Theorem 6.5. *Assume T to be piecewise expanding on the interval X, C^2 on the intervals (d_i, d_{i+1}), where $0 = d_1 < d_2 < \cdots < d_n = 1$. Then its transfer operator on measures of bounded variation satisfies for each N*

$$\|L^N\mu\|_{BV} \leqslant \lambda^N\|\mu\|_{BV} + B\|\mu\|_{L^1},$$

where

$$\lambda \leqslant 2 \cdot \left\| \frac{1}{T'} \right\|_\infty, \qquad B \leqslant \frac{2}{\min(d_i, d_{i+1})} + 2 \cdot \left\| \frac{T''}{(T')^2} \right\|_\infty.$$

For proof, see [GN14, theorem 5.2 and remark 5.3].

From now on, L will be assumed to be an operator obtained considering a convex combination $L = \sum_{i=1}^k p_i L_i$, and satisfying

$$\|L^N\mu\|_{BV} \leqslant \lambda^N\|\mu\|_{BV} + B\|\mu\|_{L^1}.$$

Remark 6.6. An invariant probability measure of an operator satisfying such a Lasota–Yorke inequality has bounded variation and satisfies $\|\mu\|_{BV} \leqslant B$.

We can now describe the approximation strategy. To obtain an estimate in L^1 we can take as approximation operator π_δ the conditional expectation operator with respect to a uniform partition F_δ of X in intervals of size δ:

$$\pi_\delta(\mu) = \mathbf{E}(\mu|F_\delta).$$

The approximated operator $L_\delta = \pi_\delta L \pi_\delta$ is known as the *Ulam approximation* of L.

Proposition 6.7. *If L_δ is defined as above then*

$$\|L_\delta - L\|_{BV\to L^1} \leqslant 2\delta.$$

For proof, we refer to lemma 5.5 of [GN14] which proves the inequality for the transfer operator of a dynamical system (as each of the L_i is). Thus being L obtained as a convex combination we have

$$\|L_\delta - L\|_{BV\to L^1} \leqslant \sum_i p_i \|L_{i,\delta} - L_i\|_{BV\to L^1} \leqslant \sum_i p_i 2\delta = 2\delta.$$

As observed in remark 6.6 we can bound from above the bounded variation norm of μ. This permits us to obtain item 1 of theorem 3.1 by using the proposition above.

Remark 6.8. As pointed out in [BB11, GN14], similar results hold for the pair of norms Lip and L^∞, in the place of the pair BV, L^1. It is hence possible to obtain similarly a rigorous computation of the invariant density in the L^∞ norm, using essentially the same strategy. The same extension can also be done in the IFS case.

7. Some remarks on implementation

The topics in this section apply equally to the experiments in sections 8 and 10. The code used in our experiments is available at

http://im.ufrj.br/~maurizio.monge/wordpress/rigorous_computation_dyn/.

To certify the numerics, we use the interval arithmetics libraries that are available through the SAGE Mathematics Software [Dev15].

The assembly of the matrix in the piecewise expanding case is done by using interval arithmetics Newton methods while the estimates for the contraction rate are made with double precision arithmetics with rigorous (and conservative) bounds on the iteration error.

In the contracting case the Newton method is not necessary, and we can rigorously assemble the approximated operator directly, using interval arithmetic to keep track of possible numerical errors.

8. Implementation in some piecewise expanding examples

Let us consider the smooth dynamical systems on [0, 1] given by

$$T_1(x) = 4x + 0.01 \cdot \sin(16\pi x), \qquad T_2(x) = 5x + 0.03 \cdot \sin(5\pi x).$$

Then in our case we have (not iterated) Lasota–Yorke constants that are

$$\begin{bmatrix} \lambda(T_1) = 0.333924 \\ B(T_1) = 11.26927 \end{bmatrix} \qquad \begin{bmatrix} \lambda(T_2) = 0.246455 \\ B(T_2) = 1.798453 \end{bmatrix}$$

and consider the systems defined via the maps T_1, T_2 where the probabilities are selected putting

$$p_1 = 0.1, 0.3, 0.5, 0.7, 0.9$$

and $p_2 = 1 - p_1$. The constants λ, B we obtain by theorem 6.5, as well as an estimate on the BV norm of the stationary measure, are respectively given by the pairs

p_1	0.1	0.3	0.5	0.7	0.9
λ	0.255 202	0.272 696	0.290 190	0.307 683	0.325 177
B	2.745 53	4.639 69	6.533 86	8.428 02	10.322 19
$\|\mu\|_{BV}$	3.686 28	6.379 31	9.205 08	12.173 66	15.296 15

We select $\delta = 2^{-16}$, obtaining

$$\|L - L_\delta\|_{BV \to L^1} \leqslant 2^{-15}$$

from proposition 6.7. We prove computationally that the Ulam matrices of T_1, T_2 contract to $\alpha = \frac{1}{2}$ in 9 and 8 steps respectively, and similarly compute the contracting rate for the IFS transfer operator. Considering that in theorem 3.1 all C_i are 1 we can estimate the L_1 error as

$$\|\mu - \mu_\delta\|_{L^1} \leqslant \|L - L_\delta\|_{BV \to L^1} \cdot \|\mu\|_{BV} \cdot \frac{N}{1 - \alpha},$$

N being such that $\|L_\delta^N\| < \alpha$ (contraction rate). As we will show later, we can also give an error estimation depending on an *a priori* estimation of the decay time. The estimation obtained in this way is rather pessimistic, but is obtained only from the contraction time for the operators associated to T_1 and T_2, that is, skipping the more numerically intensive computation.

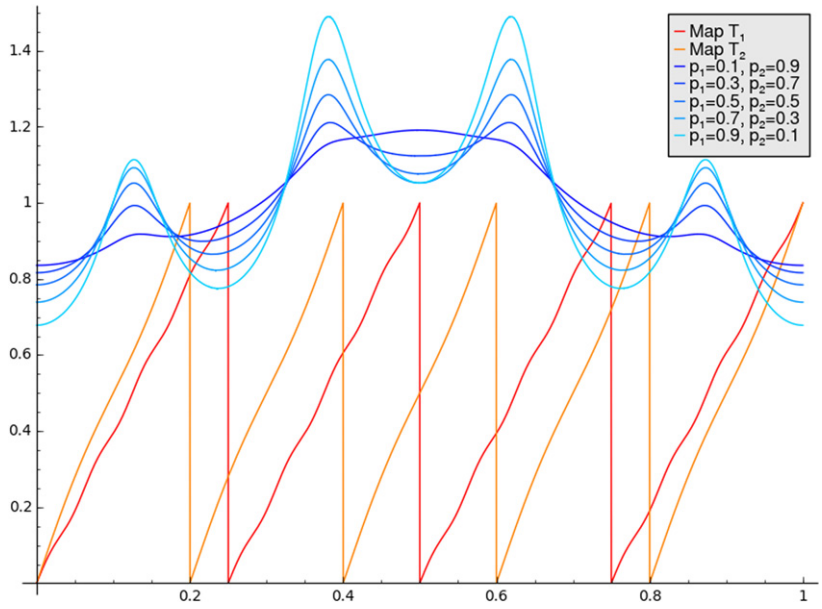

Figure 1. The stationary measures in the expanding examples.

p_1	0.1	0.3	0.5	0.7	0.9
N (contraction rate)	8	7	7	8	9
L^1error	0.001 80	0.002 72	0.003 93	0.005 94	0.008 40
Apriori$N$ (contraction rate)	34	222	2135	314	37
AprioriL^1error	0.007 66	0.0865	1.200	0.233	0.0345

In the table the contraction rate is the N such that $\|L_\delta^N\| < \alpha$, while the error is $\|\mu - \mu_{\delta,c}\|_{L^1}$ ($\mu_{\delta,c}$ being the computed approximation of μ_δ, that is, a rigorous estimate of the numerical error has been added). The *a priori* equivalents are obtained via theorem 5.2 rather than via an expensive computation; except for two central values, the *a priori* error could already be considered acceptable.

We conclude with an example of IFS formed replacing the above transformation T_1 with an irrational rotation by $\sqrt{2}$. Such a map satisfies a (trivial) Lasota–Yorke with $\lambda = 1$ and $B = 0$, but for each nontrivial value of the p_i the transfer operator of the corresponding IFS satisfied a nontrivial Lasota–Yorke (with $\lambda < 1$). Taking $\delta = 2^{-14}$, we can compute the stationary measure up to the error specified below (see figure 1).

p_1	0.2	0.4	0.6	0.8
λ	0.397 16	0.547 87	0.698 58	0.849 29
B	1.438 76	1.079 07	0.719 38	0.359 69
N (contraction rate)	8	9	12	20
L^1error	0.001 168 7	0.001 313 9	0.001 750 6	0.002 915 5

8.1. Convergence to equilibrium

We will apply here section 4's results to the case of uniformly expanding maps (see figure 2).

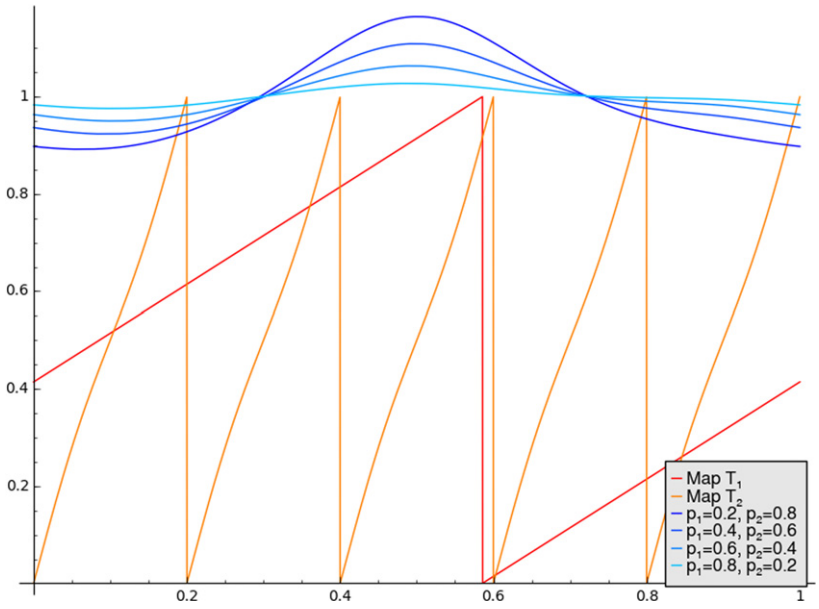

Figure 2. Examples with rotation and expanding map.

Let us compute the constants C, D appearing in the (5). Let U be a uniform estimate on the norm of $\|L_\delta^i\|_w$ and $\|L^i\|_w$. We assume a Lasota–Yorke inequality

$$\|L^i g\|_s \leqslant \lambda^i \|g\|_s + B\|g\|_w,$$

to be satisfied, and in the other hand

$$(L_\delta^n - L^n) = \sum_{i=0}^{n-1} L_\delta^{n-1-i}(L_\delta - L)L^i,$$

consequently

$$\begin{aligned}\|L^i g\|_s &\leqslant \sum_{i=0}^{n-1} \|L_\delta^{n-1-i}\|_w \cdot \|L_\delta - L\|_{s\to w} \cdot \|L^i g\|_s \\ &\leqslant U\|L_\delta - L\|_{s\to w} \cdot \sum_{i=0}^{n-1} (\lambda^i \|g\|_s + B\|g\|_w) \\ &\leqslant U\|L_\delta - L\|_{s\to w} \cdot \left(\frac{1}{1-\lambda}\|g\|_s + nB\|g\|_w\right).\end{aligned}$$

In the case of the strong norm being BV and the weak one L^1, we have $U = 1$ (because L and π_δ are contractions with respect to L^1), and $\|L_\delta - L\|_{s\to w} \leqslant 2\delta$ by proposition 6.7. Hence, we can take

$$C = \frac{2}{1-\lambda}, \qquad D = 2B.$$

We will take $\lambda_2 = 1/2$, and let $n_1 = N$ be the computed N such that $\|L_\delta^N\| \leqslant 1/2$. Consequently, the convergence to the equilibrium is controlled by the matrix

$$\mathcal{M} = \begin{pmatrix} \lambda^N & B \\ 2\delta/(1-\lambda) & 2\delta NB + 1/2 \end{pmatrix}.$$

We recall that computing the biggest eigenvalue ρ and the corresponding left eigenvector (a, b), we obtain an estimation for the asymptotical behavior of the convergence to equilibrium:

$$\|L^{n_1 k} g\|_s \leqslant \frac{\rho^k}{a}\|g\|_s, \qquad \|L^{n_1 k} g\|_w \leqslant \frac{\rho^k}{b}\|g\|_s. \tag{9}$$

Remark 8.1. Note that by (9) and the spectral radius formula ρ is a bound for the spectral radius of the operator restricted to the zero average space. Hence giving an estimation for the spectral gap.

We implemented these ideas and techniques on the first example of section 8. The following table summarizes the results of the computation, where an expression of the form $\ldots a_1 a_0 {}^{c_0 c_1 \cdots}_{b_0 b_1 \cdots}$ (a_i, b_i, c_i being digits) means that the value is rigorously contained the interval formed by the numbers with decimal expansions $\ldots a_1 a_0 b_0 b_1 \ldots$ and $\ldots a_1 a_0 c_0 c_1 \ldots$ (same notation of [Tuc10]). The table gives values for the ρ and (a, b) that appear in (9).

p_1	0.1	0.3	0.5	0.7	0.9
N (contraction rate)	8	7	7	8	9
ρ (biggest eigenval.)	0.500^{9}_{89}	0.5013^{8}_{7}	0.5019^{6}_{5}	0.502^{8}_{79}	0.5037^{7}_{6}
$\binom{a}{b}$ (left eigenvec.)	$\binom{0.000081^{8}_{7}}{0.999918^{3}_{2}}$	$\binom{0.000083^{8}_{7}}{0.999916^{3}_{2}}$	$\binom{0.000085^{7}_{6}}{0.999914^{4}_{3}}$	$\binom{0.000087^{7}_{6}}{0.999912^{4}_{3}}$	$\binom{0.000089^{8}_{7}}{0.999910^{3}_{2}}$

9. Contracting maps

In the case of contracting maps, the strategy is similar but the functional spaces are totally different. In fact, the transfer operator turns out not to be a contraction when applied to spaces of regular absolutely continuous measures like BV or C^1.

A space in which the transfer operator attached to a contracting map is a contraction is the dual of Lipschitz, that is the space of measures having finite norm

$$\|\mu\|_W = \sup_{\phi \in C^0(X):\mathrm{Lip}(\phi)\leqslant 1} \int_X \phi \mathrm{d}\mu,$$

where we denote by $\mathrm{Lip}(\phi)$ the Lipschitz constant of ϕ (with respect to some distance on X). We remark that μ has to be a zero-average measure for $\|\mu\|_W$ to be finite; if $\mu = \mu_1 - \mu_2$ for positive measures μ_1 and μ_2, $\|\mu\|_W$ is also known as the Wasserstein–Kantorovich distance (well known in transportation theory, and also known as Earth-moving distance) between μ_1 and μ_2.

Let T be a contraction with contraction rate α and L be the corresponding transfer operator. Then for each ϕ satisfying $\mathrm{Lip}(\phi) \leqslant 1$ we have

$$\int_X \phi(x)\mathrm{d}L\mu(x) = \int_X \phi(T(x))\mathrm{d}\mu(x)$$
$$= \alpha \int_X \frac{\phi(T(x))}{\alpha}\mathrm{d}\mu(x) \leqslant \alpha\|\mu\|_W$$

observing that

$$\mathrm{Lip}\left(\frac{\phi(T(x))}{\alpha}\right) \leqslant 1.$$

This proves that the operator L satisfies

$$\|L\mu\|_W \leqslant \alpha\|\mu\|_W \tag{10}$$

for each zero-average Borel measure $\mu \in V$.

Remark 9.1. From now on we will just assume L to be a contraction, i.e. satisfying (10). Since we just proved that the transfer operator associated to a single contracting map satisfies (10), this will also be true for the transfer operator of an IFS as its transfer operator is obtained as a convex combination of the operators.

Assume now X be a bounded domain in $\mathbb{R}^n$ equipped with the 'Manhattan' (L^1) distance $d_M(\cdot, \cdot)$ defined as

$$d_M(x, y) = \sum_{i=0}^{n} |y_i - x_i|,$$

with respect to which we will assume our maps to be contractions (and that we will use to evaluate the $\|\cdot\|_W$ norm). We will define a projection on the space of measures that is a weak contraction in the $\|\cdot\|_W$ distance.

We will discretize spatially the bounded measures in R^n working one-dimension at a time, depending on a parameter δ determining the coarseness of the discretization. Assume our domain X to be contained in a parallelepiped

$$\Pi = [P_1, Q_1] \times [P_2, Q_2] \times \ldots \times [P_n, Q_n],$$

and let us also assume for convenience that each size $Q_k - P_k$ is an integer multiple of δ, $N_k\delta$ say. We will put

$$p_{k,i} = P_k + \mathrm{i}\delta$$

for $1 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant N_k$, so that the $p_{k,i}$ are a uniform partition of the interval $[P_k, Q_k]$.

Remark 9.2. We will now define a projection operator π_k that aligns all the mass to the partition $p_{k,i}$ in the kth dimension. The mass that is not aligned is moved to the closest aligned points (with coordinates $p_{k,i}$ and $p_{k,i+1}$, say), linearly interpolated so that the closest points get more mass. That is, the measure $(\pi_k\mu)(A)$ of a set A will only depend on the slices of A that are partition-aligned (because the new measure is only concentrated on the aligned points), and each slice will have mass that is given by μ weighted with a hat function centered in a point of the grid, on the set of points that along the kth dimension are projected onto the slice.

Let $\sigma_{k,i}$ be the projection along the kth coordinate to the plane formed by points where such coordinate is equal to $p_{k,i}$:

$$\sigma_{k,i}((x_1,\ldots,x_n)) = (x_1,\ldots,x_{k-1},\, p_{k,i},\, x_{k+1},\ldots,x_n).$$

Let now

$$h_{k,i}(x) = \begin{cases} 1 - \frac{1}{\delta}|x_k - p_{k,i}| & \text{if } |x_k - p_{k,i}| \leqslant \delta, \\ 0 & \text{in any other case,} \end{cases}$$

so that this function goes linearly from the value of 1 at points with $x = p_{k,i}$ to 0 where $x = p_{k,i-1}$ or $= p_{k,i+1}$.

Recall that the pushforward of a measure μ via a function f is defined as

$$f_*(\mu)(A) = \mu(f^{-1}(A)).$$

Notice that given any measure μ on Π, the measure $\sigma_{k,i*}(h_{k,i}\mu)$ is obtained moving to the plane $x_k = p_{k,i}$ the nearby measure, weighted with the function $h_{k,i}$.

We define the projection as

$$\pi_{\delta,k}(\mu) = \sum_{i=0}^{N_k} \sigma_{k,i*}(h_{k,i}\mu).$$

Intuitively, $\pi_{\delta,k}$ can be viewed as the operation of moving 'sliding the kth coordinate' all the mass to the affine planes of equations $x_k = p_{k,i}$, using the functions $h_{k,i}$ to spread linearly the contribution from each point to the nearby planes.

Example 9.3. To visualize how the projection works, assume $\Pi = [0, 1] \times [0, 1]$, $\mu = \delta_{(1/3,2/5)}$ (the atomic measure centered in $(1/3, 2/5)$) and the grid size to be 1. The we have

$$\pi_{1,1}(\mu) = \frac{2}{3}\delta_{(0,2/5)} + \frac{1}{3}\delta_{(1,2/5)}, \qquad \pi_{1,2}(\mu) = \frac{3}{5}\delta_{(1/3,0)} + \frac{2}{5}\delta_{(1/3,1)},$$

and

$$\pi_{1,2}(\pi_{1,1}(\mu)) = \pi_{1,1}(\pi_{1,2}(\mu)) = \frac{2}{5}\delta_{(0,0)} + \frac{4}{15}\delta_{(0,1)} + \frac{1}{5}\delta_{(1,0)} + \frac{2}{15}\delta_{(1,1)}.$$

We put $\pi_\delta = \pi_{\delta,1} \cdots \pi_{\delta,n}$ (these operators obviously commute), and such π_δ can be easily described as

$$\pi_\delta \mu = \sum_{p=(p_{1,i_1},\ldots,p_{n,i_n})} \delta_p \cdot \int h_p \mathrm{d}\mu$$

where for such given point p of the grid δ_p is the atomic measure in p and

$$h_p = \prod_{j=1}^{n} h_{j,p_{j,i_j}}.$$

We prove the following proposition.

Proposition 9.4. *If* $\|\mu\|_W \leqslant 1$, *then* $\|\pi_{\delta,k}\mu\|_W \leqslant 1$.

Proof. We need to prove that $\int \phi \mathrm{d}\pi_{\delta,k}\mu \leqslant 1$ for each admissible function ϕ. We will describe a suitable linearization $\tilde{\phi}$ of ϕ that on one hand will satisfy

$$\int \phi \mathrm{d}\pi_{\delta,k}\mu = \int \tilde{\phi}\mathrm{d}\pi_{\delta,k}\mu, \tag{11}$$

and on the other hand will have Lipschitz constant $\leqslant 1$ and satisfy

$$\int \tilde{\phi}\mathrm{d}\pi_{\delta,k}\mu = \int \tilde{\phi}\mathrm{d}\mu. \tag{12}$$

Given ϕ, we put

$$\tilde{\phi}(x) = \sum_{i=0}^{N_k} h_{k,i}(x)\phi((x_1,\ldots,x_{k-1}, p_{k,i}, x_{k+1},\ldots,x_n)).$$

On the points x such that x_k is equal to some $p_{k,i}$ the $\tilde{\phi}$ is equal to ϕ, so (11) is clearly satisfied.

To prove (12), we can just check it on the atomic μ that are δ_y for some $y = (y_1,\ldots,y_n)$ and reason by density. Note that

$$\pi_{\delta,k}\delta_y = \sum_{i=0}^{N_k} h_{k,i}(y)\delta_{(y_1,\ldots,y_{k-1},p_{k,i},y_{k+1},\ldots,y_n)}, \tag{13}$$

so the lhs of (12) turns out to be equal to

$$\tilde{\phi}(y) = \int\int \tilde{\phi}\mathrm{d}\delta_y.$$

It remains to prove that $\tilde{\phi}$ also has Lipschitz constant $\leqslant 1$ (with respect to the Manhattan distance), but let y, z be two points and put $w = (y_1,\ldots,y_{k-1}, z_k, y_{k+1},\ldots,y_n)$. We have

$$|\tilde{\phi}(y) - \tilde{\phi}(z)| \leqslant |\tilde{\phi}(y) - \tilde{\phi}(w)| + |\tilde{\phi}(w) - \tilde{\phi}(z)|$$

and note that on the segment from y to w $\tilde{\phi}$ is piecewise linear with slope $\leqslant 1$, while $\tilde{\phi}(w) - \tilde{\phi}(z)$ is obtained as convex combination of quantities that are all $\leqslant d_M(w, z)$. Consequently

$$|\tilde{\phi}(y) - \tilde{\phi}(z)| \leqslant d_M(y, z).$$

We conclude that $\int \phi \mathrm{d}\pi_{\delta,k}\mu \leqslant 1$. □

We are left with the problem of estimating the distance between the fixed points of L and L_δ, where L is the transfer operator of the IFS (or more in general any operator satisfying $\|L\mu\|_W < \alpha\|\mu\|_W$).

Proposition 9.5. *If L and L_δ are defined as above we have*

$$\|L - L_\delta\|_{L^1\to W} \leqslant (\alpha + 1)\frac{n\delta}{2}.$$

Proof. Recall the definition of $\|\mu\|_{L^1}$ in equation (2), we start observing that

$$\|1 - \pi_{\delta,k}\|_{L^1\to W} \leqslant \frac{\delta}{2}.$$

When $\pi_{\delta,k}$ is applied to an atomic measure $\mu = \delta_y$ the mass will be split by $\pi_{\delta,k}$ in at most two atoms at distance $\delta\lambda$ and $\delta(1 - \lambda)$ and mass $1 - \lambda$ and λ, and consequently

$$\|\mu - \pi_{\delta,k}\mu\|_W \leqslant 2\delta\lambda(1 - \lambda) \leqslant \frac{\delta}{2}$$

taking the maximum over all $\lambda \in [0, 1]$. This inequality holds when μ is an atomic measure $\mu = \delta_y$, and extends to the case of μ being a finite convex combination of such measures (by

linearity of $\pi_{\delta,k}$). Such measures are dense in the space of all probability measures with respect to the $\|\cdot\|_W$ distance, hence this inequality holds for all probabilities measures μ.

Applying this estimate for all dimensions we obtain that

$$\|1 - \pi_\delta\|_{L^1 \to W} \leqslant \frac{n\delta}{2},$$

and it follows that

$$\begin{aligned}\|L - L_\delta\|_{L^1 \to W} &\leqslant \|L(1 - \pi_\delta)\| + \|(\pi_\delta - 1)L\pi_\delta\| \\ &\leqslant \|L\|_W \cdot \|1 - \pi_\delta\|_{L^1 \to W} + \|1 - \pi_\delta\|_{L^1 \to W} \cdot \|L\pi_\delta\|_{L^1} \\ &\leqslant (\alpha + 1)\frac{n\delta}{2}.\end{aligned}$$

□

Since an invariant probability measure has L^1 norm equal to 1, we have that

$$\|L\mu - L_\delta\mu\|_W \leqslant (\alpha + 1)\frac{n\delta}{2}, \tag{14}$$

and as consequence of corollay 3.2 we have

$$\|\mu - \mu_\delta\|_W \leqslant \frac{(1 + \alpha)n\delta}{2(1 - \alpha)}.$$

This gives an estimation that can be applied be applied to obtain 1 of theorem 3.1, and is our main ingredient in the estimation of the error for the approximation of the stationary measure in this kind of systems.

10. An example in the contracting case

In the contactive case each map has an approximated transfer operator $L_\delta = \pi_\delta L \pi_\delta$ (represented as a matrix) that can be computed very easily. We take the image of a δ_x for x in the grid (an atom in x) via the map and approximate the $\delta_{T(x)}$ obtained (which will not be aligned to the grid) to the measure $\pi_\delta(\delta_{T(x)})$ supported on the grid using equation (13).

We take all δ_x for x in the grid as a basis of the finite dimensional space of measures supported on the grid, and being $L_\delta(\delta_x) = \pi_\delta(\delta_{T(x)})$ we obtained the expression of $L_\delta(\delta_x)$ as combination of elements of the basis. The matrix P obtained represents the operator L_δ in this basis, and the decay time of L_δ (or of P) can be estimated by the contraction rate of L via proposition 9.4.

The IFS also has a transfer operator, obtained as the linear combination of the transfer operator of the maps, and is also contracting being a convex combination of contracting operators, (its decay time can obtained as combination of the decay times of the maps). The obtained matrix is then iterated to approximate the fixed point of L_δ, the goodness of the approximation is then a consequence of its proven decay time.

The informal description of the algorithm is the same as explained in section 3, with the difference that the decay time N is estimated directly from the contraction rate of the operator L. The matrix P representing (an approximation) of L_δ is built as explained in the beginning of the section.

We made an example in the following case: the maps $T_1, \ldots, T_4$ are defined on the square $[0, 1] \times [0, 1]$ as

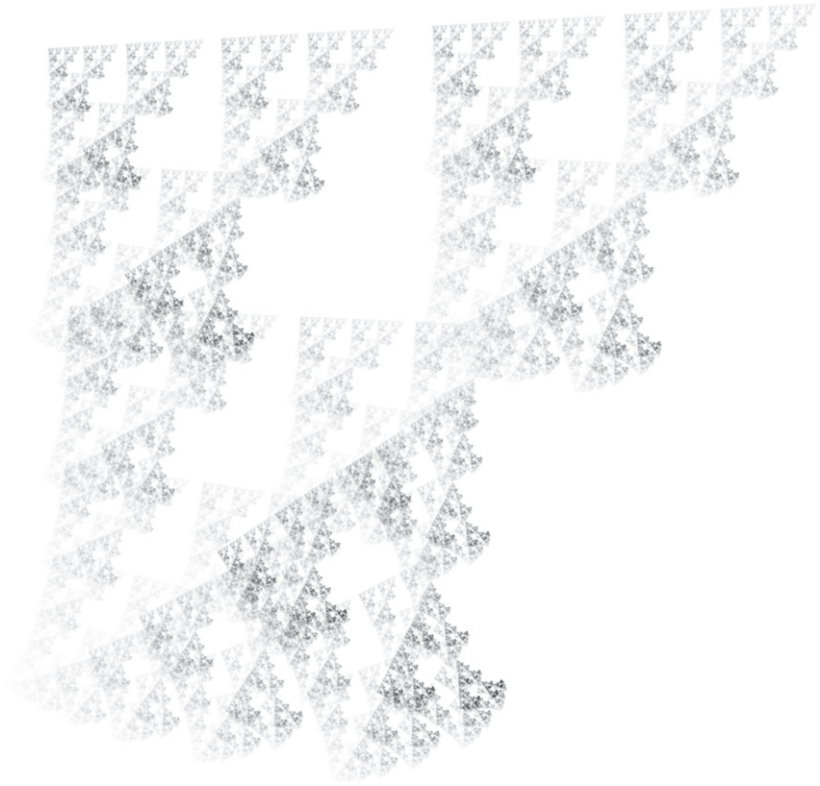

Figure 3. The stationary measure of the contracting example.

- $T1$: scaling by 0.4 around (0.6, 0.2) with rotation by $\pi/6$,
- $T2$: scaling by 0.6 around (0.05, 0.2) with rotation by $-\pi/30$,
- $T3$: scaling by 0.5 around (0.95, 0.95),
- $T4$: scaling by 0.45 around (0.1, 0.9).

We took the probabilities

$$p_1 = 0.18, \quad p_2 = 0.22, \quad p_3 = 0.3, \quad p_4 = 0.3,$$

and a grid of size $2^{10} \times 2^{10}$, so that $\delta = 2^{-10}$. It turns out that the contraction rate α is at most 0.659 430, so the error can be estimated by

$$\|\mu - \mu_\delta\|_W \leqslant \frac{(1+\alpha)n\delta}{2(1-\alpha)} \leqslant 0.004\,7583.$$

Here is an image of the computed invariant measure (see figure 3).

Remark 10.1. The computation of the contraction rate as done in section 8 for the expanding case is not necessary here, as we already know that the transfer operator is a contraction with respect to $\|\cdot\|_W$ as proved at the beginning of section 9 (with explicit constants).

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