Journal of Statistical Physics (2023) 190:22
https://doi.org/10.1007/s10955-022-03041-y

How Does Noise Induce Order?

Isaia Nisoli[1,2,3]

Received: 25 September 2021 / Accepted: 16 November 2022

Abstract
In this paper we present a general result with an easily checkable condition that ensures a transition from chaotic regime to regular regime in random dynamical systems with additive noise. We show how this result applies to a prototypical family of nonuniformly expanding one dimensional dynamical systems, showing the main mathematical phenomenon behind Noise-induced Order.

Keywords Noise-induced order · Random dynamical systems · Non-uniformly hyperbolic · Unimodal maps · Lyapunov exponents

Mathematics Subject Classification Primary 37H05; Secondary 37C30 · 37A30 · 37D25 · 37H15

1 Introduction

This article deals with the behavior of one dimensional nonuniformly hyperbolic systems with random additive noise.

A nonuniformly hyperbolic one dimensional dynamical system is a dynamical system in which expansion and contraction coexists; the behavior of such a system is a delicate balance between how often the orbits of such a system visit the expanding and contracting region. Such a system is called non uniformly expanding when the system visits more often the expanding region of the phase space.

Such balance may depend in non-trivial ways from a parameter: classical examples of unimodal maps, as the quadratic family, have a dense subset of parameters for which the deterministic dynamic presents an attracting periodic orbit, called **regular** parameters and a

---

Communicated by Peter Balint.

✉ Isaia Nisoli
nisoli@im.ufrj.br; isaia.nisoli@es.hokudai.ac.jp

1 Instituto de Matemática, Universidade Federal do Rio de Janeiro, Av. Athos da Silveira Ramos, 149-Edifício do Centro de Tecnologia, Bloco C (Térreo), Cidade Universitária, Rio de Janeiro, Brazil

2 Department of Mathematics, Hokkaido University, N10 W8, Kita-ku, Sapporo 001-0010, Japan

3 RIES, Hokkaido University, N20 W10, Kita-ku, Sapporo 001-0020, Japan

Published online: 28 November 2022

positive Lebesgue measure Cantor set of parameters for which the dynamic shows chaotic behavior called **stochastic** parameters.

In this paper we will study a generalization of the quadratic family, allowing the order of the critical point to vary, the family

$$T_{\alpha,\beta}(x) = 1 - 2\beta|x|^{\alpha};$$

these are symmetric unimodal maps, defined on $[-1, 1]$, for $\alpha \in [2, +\infty)$, $\beta \in (0, 1]$.

We will study the behavior under iterations of these maps with the addition of a random noise at each iteration step, i.e.,

$$X_{n+1} = T_{\alpha,\beta}(X_n) + \Omega_{\xi},$$

where Ω_{ξ} is a random variable which takes values in $[-\xi, \xi]$ with density

$$\rho_{\xi}(x) = \frac{1}{\xi}\rho\left(\frac{x}{\xi}\right),$$

where ρ is a positive BV density on $[-1, 1]$. We call ξ the **amplitude** of the noise; we denote the points of the orbits with a capital X to stress the fact that they are random variables. This is called a **random dynamical system with additive noise of amplitude** ξ.

We will show that when $\beta = 1$ and α is bigger than a computable constant $\tilde{\alpha} > 2.67835$[1] as the noise amplitude increases, the system transitions from a chaotic behavior to an ordered behavior; this transition is measured quantitatively by a transition of the Lyapunov exponent associated to the stationary measure from positive to negative.

This surprising phenomenon is called in the literature Noise Induced Order and was first observed in numerical simulations of a model of the Belosouv–Zhabotinsky reaction [16], called the BZ map; a proof of its existence for the BZ map was given recently in [11].

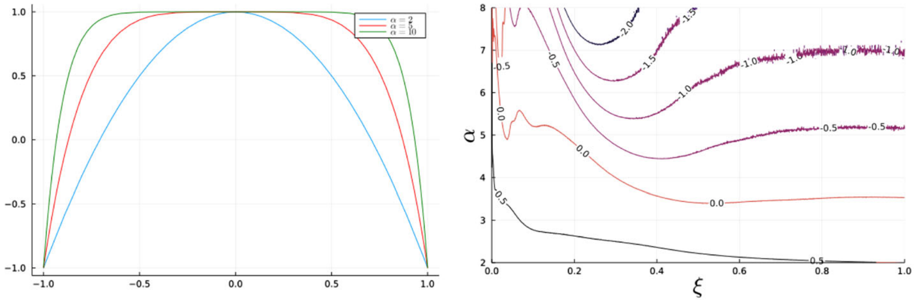
In this paper we show the main mechanism behind this phenomenon; the presence of noise changes the statistical properties of the dynamical system, in particular, if we start with a non-uniformly expanding map, adding noise may break the delicate balance between expansion and contraction, and the average long term behavior changes from expanding to contracting.

Many Noise Induced phenomena [8, 21] are of strong interest for the applied mathematical community and in general for applied sciences but until recently they have not woken the interest of the dynamical system community. Important results have been reached in [6, 7], in the study of the Henon and the Standard map with noise.

In Fig. 1 we have a plot of some numerical experiments on the family $T_{\alpha,\beta}$, where fixed $\beta = 1$, for each exponent α (in the vertical axis) and noise amplitude ξ (in the horizontal axis) we compute 200 orbits of length 10,000, each one with a randomly chosen starting point and different random realizations of the noise, and compute the average of $\ln(|T'(x)|)$ with respect to the length of the orbit, and take the average of these Birkhoff averages; the rationale behind this is that supposing that the simulated system satisfies some type of Central Limit Theorem the mean of the finite time Birkhoff averages of all these orbits is a a better estimator of the Lyapunov exponent than the average along an individual orbit.

This plot hints that Noise Induced Order may be present in the family $T_{\alpha,\beta}$; on the left side of the plot, which presents the value of the estimator when the noise amplitude is 0 the estimator is positive. On the right side of the plot, for noise amplitude 1.0, we can see that if the order of the critical point is big enough the estimator is negative.

[1] the value of $\tilde{\alpha}$ is contained in $[2.67834, 2.67835]$, therefore, our result does not apply to the case $\alpha = 2$, the quadratic family



(A) $T_{\alpha,1}$ for $\alpha = 2, 5, 10$

(B) Lyapunov exponent in function of noise amplitude ξ and order of the singularity α

Fig. 1 The family $T_{\alpha,1}$ and estimate of its Lyapunov exponent as α and the noise size ξ vary

More complex behavior can be conjectured from this plot: there are values of α for which we can observe multiple sign changes, but the results in this paper only allow us to prove the existence of one transition. The existence of multiple transitions could be proved by using Computer Aided Tools as the ones used in [9, 11].

The article [16] has been highly influential in the applied sciences; we think that a sufficient condition for the existence of this phenomena is extremely interesting both for the researchers in dynamical systems, since we present a wide family of examples whose deterministic behavior and behavior under the action of noise are different, and for the applied scientists, since this sufficient condition is easily checkable.

Our paper shows that Noise Induced Order is strongly linked with nonuniformly hyperbolic dynamics and that the existence of this kind of phenomena stresses the importance of the study of random dynamical systems beyond stochastic stability. We think that our paper will contribute to show the richness of the behavior of dynamical systems with additive noise.

Our choice of the title is a direct answer to [16]; indeed, from the results of [11] and the results in the current paper, we can assert that we found the main mechanism for Noise Induced Order in the 1-dimensional case. Please remark that to apply the techniques in the present article to the BZ map from [11] we need a computer assisted step: the BZ map does not fit in our framework to study stochastic stability for nonuniformly hyperbolic maps but our argument works once positive Lyapunov exponent and contraction of the space of average 0 functions is proved for a small noise amplitudes, which is the difficult part of [11] and the main computer aided estimate.

1.1 Statement of the Results

In this paper we prove that, under some assumptions, for all noise amplitudes $\xi > \xi_0$ the random dynamical system has a unique ergodic absolutely continuous stationary measure μ_ξ.

The Birkhoff Ergodic Theorem for Random Dynamical Systems tells us that, for a fixed noise size ξ, for μ_ξ-a.e. initial condition x_0 and for almost all noise realizations we have that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n} \ln(|T'(X_i)|) = \int \ln(|T'|)d\mu_\xi.$$

As ξ varies we are interested in the behavior of the Lyapunov exponent as a function of noise amplitude; remark that in the next formula, as ξ varies μ_ξ is varying:

$$\lambda(\xi) = \int \ln(|T'|) d\mu_\xi;$$

as in [11], we define Noise Induced Order as follows.

Definition 1.1.1 We say that a system exhibits **Noise Induced Order** if there exist $0 < \xi_1 < \xi_2$ such that for all $\xi \geq \xi_1$ the system has a unique stationary measure with density f_ξ and the Lyapunov exponent of the stationary measure transitions continuously from positive to negative, i.e., $\lambda(\xi_1) > 0$, $\lambda(\xi_2) < 0$.

Remark 1.1.2 There is an ongoing discussion in the community on the "right" definition of Noise Induced Order; in [16] are indicated:

- sharpening of power spectrum,
- abrupt decrease of entropy,
- appearance of negative Lyapunov number,
- localization of orbit.

In [11] the existence of a transition for the Lyapunov exponent from positive to negative was used as a definition of Noise Induced Order. The continuity argument in the present paper shows that there exists a "big" noise amplitude such that for all noise amplitudes bigger than this given noise amplitude, the Lyapunov exponent is negative.

Remark 1.1.3 The definition allows a deterministic map with negative Lyapunov exponent to show noise induced order: a regular parameter under the action of noise may show a transition to positive Lyapunov exponent for a small noise amplitude and a negative Lyapunov exponent for a larger noise amplitude.

Our method to prove the existence of this transition is quite general and it follows from two simple observations: the first one is that once contraction of the space of average 0 functions in BV (also called exponential decay of correlations in BV) is proved for a noise size ξ_0, the Lyapunov exponent is continuous with respect to ξ for all $\xi > \xi_0$.

The second one is that as the noise amplitude grows, the density of the stationary measure becomes uniform, and therefore, the limiting behavior of the Lyapunov exponent is the average of $\ln(|T'|)$ with respect to the uniform density on $[-1, 1]$.

Many results on stochastic stability have been proved [1, 2, 4, 22] that hint on the direction that positive Lyapunov exponent may imply stochastic stability (see also the conjecture in [24]). Therefore we would like to state the following conjecture.

Conjecture 1.1.4 *Let $T : I \to I$ be a piecewise C^1 dynamical system, nonsingular with respect to Lebesgue measure, which admits a unique absolutely continuous invariant measure with positive Lyapunov exponent; if*

$$-\infty < \int_{-1}^{1} \ln(|T'|) \frac{dm}{2} < 0$$

then the associated random dynamical system with additive noise with bounded variation density presents Noise Induced Order.

Remark 1.1.5 The uniform density on $[-1, 1]$ is $1/2$, which is the reason why many $1/2$ appears in the hypothesis above and in the conditions below.

While this notation is unneccessary, we would like to state the conditions in this form, to stress the mechanism underlying the transition.

While we cannot prove this conjecture in its full generality, due to the technical difficulties involved in proving stochastic stability in a general setting, in this paper we prove the following theorem; please note that the hypothesis denoted by D are hypothesis on the deterministic system, while the hypothesis denoted by R are hypothesis on the associated random dynamical system with additive noise.

Theorem 1.1.6 *Let $T : [-1, 1] \to [-1, 1]$ be a piecewise C^1 nonsingular dynamical system such that*

D1 *admits a unique absolutely continuous invariant probability measure μ_0 with density f_0,*
D2 $\int_{-1}^{1} \ln(|T'|)d\mu_0 > 0$,
D3 $\ln(|T'|) \in L^p$ *for some* $p > 1$.

Let μ_ξ be a fixed point for L_ξ, the annealed transfer operator (Defined in 2.4.13) and let

$$\lambda(\xi) = \int_{-1}^{1} \ln(|T'|)d\mu_\xi.$$

Suppose now:

R1 *there exists ξ_0 such that $\lambda(\xi)$ is well-defined and continuous in $[0, \xi_0)$,*
R2 *there exist ξ_1 in $[0, \xi_0)$, $C > 0, \theta < 1$ such that $||P_{\xi_1}^n|_{\mathcal{U}_0}||_{BV} \le C\theta^n$, where P_{ξ_1} is the annealed Perron–Frobenius operator associated to the random dynamical system with noise size ξ_1, defined in Definition 2.4.13 and $\mathcal{U}_0$ is the subspace of BV functions with average 0 defined in Definition 2.5.6,*
R3 $-\infty < \int_{-1}^{1} \ln(|T'(x)|)dm/2 < 0$,
R4 *the noise kernel is a mother noise kernel (Definition 2.4.11).*

Then, the function $\lambda(\xi)$ is well defined for and continuous for $\xi \ge 0$ and the map T exhibits Noise Induced Order.

The proof of the theorem is found in Sect. 3.

We will try to give an intuition behind the hypothesis for this Theorem and our proof. Hypothesis D1 and D2 are telling us that the original system has positive Lyapunov exponent; the deterministic system is chaotic, D3 is a mild regularity assumption. Hypothesis R1 follows from the stochastic stability and tells us that Lyapunov exponent is continuous in a small neighborhood of 0. Hypothesis R2 kickstarts our continuity argument; remark that even if the underlying dynamical system has subexponential decay of correlations, this hypothesis may be satisfied, due to the smoothing properties of noise. Hypothesis R3 is an hypothesis on the behavior of the system when the noise is big; as the noise size increases the noise moves the random orbits uniformly inside the system, so the Lyapunov exponent along random orbits is negative. Hypothesis R4, the fact that the noise density is a mother noise kernel can be intuitively understood as the fact that, as the noise amplitude increases, the support of the noise contains the support of smaller amplitude noises.

The consequences for the family $T_{\alpha,\beta}$ are summarized in the following theorem and are proved in Sect. 4

Theorem 1.1.7 *Let* $T : [-1, 1] \to [-1, 1]$ *be of the form*

$$T_{\alpha,\beta}(x) = 1 - 2\beta|x|^{\alpha}.$$

For all $\alpha > \tilde{\alpha} > 2.67835$ *there exists an* $\epsilon(\alpha)$ *such that for all* $\beta \in (1 - \epsilon(\alpha), 1]$ *the map* $T_{\alpha,\beta}$ *exhibits Noise Induced Order.*

1.2 Structure of the Paper

We start in Sect. 2 where we introduce the annealed transfer operator, prove that contraction of the space of average 0 function for a noise amplitude implies contraction for all bigger noise amplitudes and prove that the Birkhoff averages of $L^1(m)$ observables are continuous with respect to noise size and the underlying dynamics once contraction of average 0 functions is established. Section 3 is a small section, where we present the proof of Theorem 1.1.6. Section 4 studies the family $T_{\alpha,\beta}$, by producing results on its stochastic parameters, stochastic stability and showing the conditions on α and β that imply that maps in the family present noise induced order.

2 The Annealed Transfer Operator

2.1 Generalities on the Involved Functional Spaces

Definition 2.1.1 Let $[a, b] \subset \mathbb{R}$ be an interval endowed with the Lebesgue measure m; we denote by $L^r([a, b])$ the Banach space of real valued functions such that

$$||f||_{L^r([a,b])} := \sqrt[r]{\int_a^b |f|^r dm} < +\infty.$$

We will drop the interval from the notation when clear from the context. Of particular interest for us is $L^1([-1, 1])$.

We define $L^{\infty}([a, b])$ to be the Banach space of real valued functions such that

$$||f||_{L^{\infty}([a,b])} := \text{esssup}_{x\in[a,b]}|f(x)| < +\infty,$$

where the essential supremum is the smallest real number a such that $|f(x)| \leq a$ for m-almost every x in $[a, b]$.

Definition 2.1.2 We will call a **density** a nonnegative function $f \in L^1([a, b])$ such that

$$\int_a^b f dm = 1$$

Lemma 2.1.3 *If* f *is a density, then*

$$||f||_{L^1([a,b])} = 1.$$

Proof This follows from the definition, since f is nonnegative

$$\int_a^b |f| dm = \int_a^b f dm = 1.$$

□

Definition 2.1.4 Let $\phi : [a, b] \to \mathbb{R}$ be a real valued function on $[a, b]$, we define the **variation** of ϕ on $[a, b]$ as

$$\mathrm{Var}_{[a,b]}(\phi) = \sup_{\mathcal{P}} \sum_i |\phi(x_{i+1}) - \phi(x_i)|$$

where $\mathcal{P}$ is any partition of $[a, b]$ with endpoints x_i. If the variation of ϕ is finite, we say that ϕ is a function of bounded variation on $[a, b]$. The functions of bounded variation on $[a, b]$ are a Banach space when equipped with the norm

$$||\phi||_{BV([a,b])} := ||\phi||_{L^1([a,b])} + \mathrm{Var}_{[a,b]}(\phi).$$

When the domain $[a, b]$ is clear, we will drop the subscript.

In the following lemma, we need to pay some attention on the domain of definitions of the functions and to the support of measures: the convolution is defined in general for functions defined on the real line, while we speak of functions which are L^1 or bounded variation on intervals.

Definition 2.1.5 Denote by χ_X is the characteristic function of the set X.

In the following we will denote by

$$\hat{\phi} = \phi \cdot \chi_{[-\xi,\xi]}$$

and by

$$\hat{f} = f \cdot \chi_{[-1,1]}$$

the functions that extend by 0 outside their intervals of definition the functions ϕ and f respectively.

Given a probability measure μ on $[-1, 1]$ we define its extension $\hat{\mu}$ on $\mathbb{R}$ as the unique measure $\hat{\mu}$ on $\mathbb{R}$ such that

$$\hat{\mu}(A) = \mu(A \cap [-1, 1])$$

for all A measurable in $\mathbb{R}$.

Lemma 2.1.6 *The following are true:*

(1) $Var_{[a,b]}(\hat{\phi}) \le Var_{[-\xi,\xi]}(\phi) + 2\sup_{[-\xi,\xi]} |\phi(x)| \le 3||\phi||_{BV([-\xi,\xi])}$ *for all interval* $[a, b]$ *that contains* $[-\xi, \xi]$,
(2) $||\hat{f}||_{L^1([c,d])} = ||f||_{L^1([-1,1])}$ *for all interval* $[c, d]$ *that contains* $[-1, 1]$,
(3) $\hat{\mu}([-1 - \xi, 1 + \xi]) = 1$ *for all* ξ.

Proof Both follow from the respective definitions. We first prove item (1); let $\{x_1 = a, \ldots, x_n = b\}$ be a partition of $[a, b]$, without loss of generality we can suppose that $x_l = -\xi$, $x_{l+k} = \xi$.

Then

$$\sum_{i=1}^{n} |\hat{\phi}(x_{i+1}) - \hat{\phi}(x_i)| = |\hat{\phi}(x_l)| + \sum_{i=l}^{l+k-1} |\hat{\phi}(x_{i+1}) - \hat{\phi}(x_i)| + |\hat{\phi}(x_{l+k})|$$

$$= |\phi(x_l)| + \sum_{i=l}^{l+k-1} |\phi(x_{i+1}) - \phi(x_i)| + |\phi(x_{l+k})| \le \mathrm{Var}_{[-\xi,\xi]}(\phi) + 2 \sup_{[-\xi,\xi]} |\phi(x)|.$$

We prove now item (2):

$$\int_c^d |\hat{f}|dm = \int_c^d |f \cdot \chi_{[-1,1]}| = \int_{-1}^1 |f|dm = ||f||_{L^1([-1,1])}.$$

Finally item (3)

$$\hat{\mu}([-1-\xi, 1+\xi]) = \mu([-1-\xi, 1+\xi] \cap [-1,1]) = \mu([-1,1]) = 1.$$

□

2.2 Regularization Properties of Convolution on Measures

In this subsection we define what is the convolution of a measure with respect to a bounded variation function and prove some regularization properties of this operator; the most important is that convolution of a measure with a bounded variation functions is a measure which is absolutely continuous with respect to Lebesgue.

Definition 2.2.1 Let μ be any probability measure in $[-1, 1]$, and let ρ be a bounded variation function on $[-\xi, \xi]$ with $\int_{-\xi}^{\xi} \rho = 1$; their convolution is the unique probability measure $\hat{\rho} * \hat{\mu}$ on $\mathbb{R}$ such that

$$\hat{\rho} * \hat{\mu}(A) = \int_{[-\xi,\xi]} \hat{\rho}(y)\hat{\mu}(A-y)dm(y),$$

where $A - y$ to denote the set $\{x - y \mid x \in A\}$.

Lemma 2.2.2 *The following properties of $\hat{\rho} * \hat{\mu}$ are true:*

(1) $\hat{\rho} * \hat{\mu}([-1-\xi, 1+\xi]) = 1$,
(2) *if $\mu = \delta_p$, the Dirac-δ measure at $p \in [-1, 1]$ we have that*

$$\hat{\rho} * \hat{\delta_p} = \hat{\rho}(x - p) \cdot m(x);$$

in particular, $\hat{\rho} * \hat{\delta_p}$ is absolutely continuous with respect to Lebesgue,
(3) *if $\mu = f dm$ then $\hat{\rho} * \hat{\mu}$ has density $\hat{\rho} * \hat{f}$.*

Proof Item (1) follows from the definition and Item (3) in Lemma 2.1.6:

$$\hat{\rho} * \hat{\mu}([-1,1]) = \int_{[-\xi,\xi]} \hat{\rho}(y)\hat{\mu}([-1-\xi, 1+\xi])dm(y)$$
$$= \mu([-1,1]) \int_{[-\xi,\xi]} \hat{\rho}(y)dm(y) = 1.$$

Item (2) follows from the definition, recalling that $\delta_p(A) = \chi_A(p)$:

$$\hat{\rho} * \hat{\delta_p}(A) = \int_{[-\xi,\xi]} \hat{\rho}(y)\hat{\delta_p}(A-y)dm(y) = \int_{[-\xi,\xi]} \hat{\rho}(y)\chi_{A-y}(p)\chi_{[-1,1]}(p)dm(y)$$
$$= \int_{[-1-\xi,1+\xi]} \chi_{[-\xi,\xi]}(y)\rho(y)\chi_A(p+y)dm(y)$$
$$= \int_{[-1-\xi,1+\xi]} \chi_{[-\xi,\xi]}(z-p)\rho(z-p)\chi_A(z)dm(x) = \int_A \hat{\rho}(z-p)dm(z),$$

where in the last line we used the change of variables $z = p + y$.
Item(3) follows from the definition. □

We prove now a general result on sequences of absolutely continuous probability measures with uniformly bounded densities.

Lemma 2.2.3 *Let $\mu_n = f_n dm$, $n \in \mathbb{N}$ be a sequence of absolutely continuous probability measures such that:*

- *$f_n \in L^\infty(m)$ for all n*
- *there exists an interval $[a, b]$ such that $\mu_n(\mathbb{R} \setminus [a, b]) = 0$, for all n,*
- *μ_n converges weakly to μ,*
- *there exists $M > 0$ such that $||f_n||_\infty \leq M$ for all $x \in [a, b]$, $n \in \mathbb{N}$.*

then μ is an absolutely continuous probability measure, with $\mu(\mathbb{R} \setminus [a, b]) = 0$.

Proof By Portmanteau theorem, weak convergence of μ_n to μ implies that for all open sets A

$$\mu(A) \leq \liminf \mu_n(A).$$

This implies that

$$\mu(\mathbb{R} \setminus [a, b]) \leq \liminf \mu_n(\mathbb{R} \setminus [a, b]) = 0.$$

The fact that μ is a probability measure follows from definition of weak convergence.

Suppose now B is a measurable set; we claim that if m is the Lebesgue measure $m(B) = 0$ implies $\mu(B) = 0$.

Let B measurable, without loss of generality we can suppose $B \subseteq [a, b]$, and let A be any open set containing B; by weak convergence, Portmanteau theorem and the fact that $||f_n||_\infty \leq M$ for all n we have that:

$$\mu(B) \leq \mu(A) \leq \liminf \mu_n(A) = \liminf \int_A f_n dm \leq M \cdot m(A).$$

We recall that the Lebesgue measure m on $\mathbb{R}$ is outer regular, i.e., for all measurable sets B we have that $m(B) = \inf\{m(A) \mid A \text{ open}, B \subseteq A\}$; taking the inf over all open sets A containing B on the right side of the inequality above implies that $\mu(B) \leq M \cdot m(B)$ and absolute continuity of μ. □

Remark 2.2.4 The hypothesis that the f_n are uniformly bounded is fundamental in the proof above, and the theorem is false if it is not satisfied. An example is the sequence $f_n = 1/(2\xi)\chi_{[-\xi,\xi]} \cdot m$ which converges weakly to δ_0.

Remark 2.2.5 This Lemma is a folklore result in measure theory [19]; in the provided link, different proofs and a discussion of the result are provided.

We sketch another proof, found at the provided link, with a more functional analytic approach: by classical results $C^\infty([a, b])$ is dense in $L^2([a, b])$. We define a sequence of functionals $T_n(g) : C^\infty([a, b]) \to \mathbb{R}$ by $T_n(g) = \int g d\mu_n$; by weak convergence, for each $g \in C^\infty([a, b])$ we can define $T(g) := \lim_{n\to+\infty} \int g d\mu_n = \int g d\mu$.

We show now that, since the f_n are uniformly bounded T can be extended to a functional $T : L^2([a, b]) \to \mathbb{R}$; this follows from the Cauchy–Schwarz inequality, since

$$|T_n(g)| = |\int g f_n dm| \leq ||g||_{L^2} ||f_n||_{L^2} \leq ||g||_{L^2} \sqrt{b-a} \cdot M,$$

since the bound is uniform in n the functional $T(g)$ can be extended to a bounded linear functional on L^2. By Riesz representation theorem, there exists an $f \in L^2([a,b])$ such that

$$T(g) = \int g \cdot f dm,$$

which implies that $\mu = f \cdot m$ is absolutely continuos.

Lemma 2.2.6 *Let μ_n be a sequence of probability measures on $[-1,1]$ weakly converging to μ. Then $\hat{\rho} * \hat{\mu_n}$ converges weakly to $\hat{\rho} * \hat{\mu}$.*

Proof By definition of weak convergence we have that for all ϕ Lipschitz on $[-1,1]$ we have that

$$\lim_{n\to+\infty} \int_{-1}^{1} \phi d\mu_n = \int_{-1}^{1} \phi d\mu.$$

Let ϕ be a Lipschitz continuous function on $[-1-\xi, 1+\xi]$, then, since $\int_{-\xi}^{\xi} \rho dm = 1$ we have that

$$\left|\int \hat{\rho}(x+h-y)\phi(y)dm(y) - \int \hat{\rho}(x-y)\phi(y)dm(y)\right|$$
$$= \left|\int \hat{\rho}(z)\left(\phi(x+h-z) - \phi(x-z)\right)dm(z)\right| \leq L \cdot h$$

where L is the Lipschitz constant of ϕ.

Now, for each ϕ Lipschitz on $[-1-\xi, 1+\xi]$ we have

$$\int \phi(x) \int \hat{\rho}(x-y)d\hat{\mu}_n(y)dm(x) = \int\int \phi(x)\hat{\rho}(x-y)dm(x)d\hat{\mu}_n(y),$$

by the inequality above $\phi * \hat{\rho}$ is Lipschitz continuous and so is its restriction to $[-1,1]$; therefore, for each ϕ Lipschitz on $[-1-\xi, 1+\xi]$ we have

$$\lim_{n\to+\infty} \int \phi(x)d(\hat{\rho} * \hat{\mu}_n)(x) = \lim_{n\to+\infty} \int_{-1}^{1} (\hat{\rho} * \phi)(x)d\mu_n(x)$$
$$= \int_{-1}^{1} (\hat{\rho} * \phi)(x)d\mu(x) = \int_{-1-\xi}^{1+\xi} \phi(x)d(\hat{\rho} * \hat{\mu})(x).$$

□

We prove now the final result of this section, that shows that convolution with a bounded variation kernel maps probability measures into probability measures which are absolutely continuous with respect to Lebesgue.

Lemma 2.2.7 *Let μ be a probability measure in $[-1,1]$, then $\hat{\rho} * \hat{\mu}$ is a probability measure on $[-1-\xi, 1+\xi]$, absolutely continuous with respect to Lebesgue.*

Proof Recall from Definition 2.2.1 that ρ is a bounded variation function on $[-\xi, \xi]$ with $\int_{-\xi}^{\xi} \rho dm = 1$.

The proof follows from Lemma 2.2.3; let $\{-1-\xi = x_0, \ldots x_{n+1} = 1+\xi\}$ be a partition of $[-1-\xi, 1+\xi]$ such that $x_{i+1} - x_i \leq 2/n$ for all $i = 0, \ldots, n$. Let

$$\mu_n = \sum_{i=0}^{n} \mu([x_i, x_{i+1}])\delta_{p_i}$$

where $p_i = (x_{i+1}+x_i)/2$ and δ_{p_i} is the Dirac-δ measure centered at p_i. Then μ_n converges weakly to μ, and $\hat{\rho} * \hat{\mu}_n$ converges weakly to $\hat{\rho} * \hat{\mu}$ by Lemma 2.2.6.

By Lemma 2.2.2 Item (2) and linearity of convolution we have that

$$\hat{\rho} * \hat{\mu}_n = \sum_{i=0}^{n} \mu([x_i, x_{i+1}])\hat{\rho}(x - p_i) \cdot m,$$

which, for each n, is an absolutely continuous probability measure whose density is uniformly bounded, i.e.,

$$\left|\sum_{i=0}^{n} \mu([x_i, x_{i+1}])\hat{\rho}(x - p_i)\right| \leq ||\rho||_{L^\infty([-\xi,\xi])} \sum_{i=0}^{n} \mu([x_i, x_{i+1}]) \leq ||\rho||_{BV}.$$

Then, by Lemma 2.2.3 we have that $\hat{\rho} * \hat{\mu}$ is an absolutely continuous probability measure. □

2.3 Regularization Properties of Convolution on Densities

Lemma 2.3.1 *Let $f \in L^1([-1, 1])$ and let ϕ be a bounded variation function on $[-\xi, \xi]$; then, their convolution*

$$\hat{\phi} * \hat{f}(x) := \int_{-\infty}^{\infty} \hat{\phi}(x - y)\hat{f}(y)dy$$

is a bounded variation function with support in $[-1 - \xi, 1 + \xi]$, such that

$$Var_{[-1-\xi,1+\xi]}(\hat{\phi} * \hat{f}) \leq \left(Var_{[-\xi,\xi]}(\phi) + 2 \sup_{[-\xi,\xi]} |\phi(x)|\right) ||f||_{L^1([-1,1])}.$$

Morever, if $\phi(x) \geq 0$ and $\int_{[-\xi,\xi]} \phi(x)dm(x) = 1$, then $||\hat{\phi} * \hat{f}||_{L^1([-1-\xi,1+\xi])} \leq ||f||_{L^1([-1,1])}$.

Proof Let τ_x be the translation operator on functions, i.e., $(\tau_y\hat{\phi})(x) = \hat{\phi}(x-y)$. By definition

$$\text{Var}_{[-\xi,\xi]}(\hat{\phi}) = \text{Var}_{[y-\xi,y+\xi]}(\tau_y\hat{\phi}).$$

We first remark that by definition of $\hat{\phi}$ and $\hat{f}$, their convolution $\hat{\phi} * \hat{f}$ is 0 outside $[-1 - \xi, 1 + \xi]$.

We observe now that for any partition $\mathcal{P}$ of $[-1 - \xi, 1 + \xi]$ we have that

$$\int_{-1}^{1} \sum_i |\hat{\phi}(x_i - y) - \hat{\phi}(x_{i+1} - y)||\hat{f}(y)|dy \leq \int \text{Var}_{[y-\xi,y+\xi]}(\tau_y\hat{\phi})|\hat{f}(y)|dy;$$

observing that $\tau_y\hat{\phi}$ is 0 outside of $[y - \xi, y + \xi]$, and that

$$\sum_i |\hat{\phi}(x_i - y) - \hat{\phi}(x_{i+1} - y)| \leq \text{Var}_{[y-\xi,y+\xi]}(\tau_y\hat{\phi}) \leq \text{Var}_{[\xi,\xi]}(\phi) + 2 \sup_{[-\xi,\xi]} |\phi(x)|,$$

by definition of variation.

Therefore

$$\int \sum_i |\hat{\phi}(x_i - y) - \hat{\phi}(x_{i+1} - y)||\hat{f}(y)|dy \leq \text{Var}_{[-\xi,\xi]}(\hat{\phi})||f||_{L^1([-1,1])} \tag{1}$$

remark that the fact that the left handside above is bounded will allow us to interchange the summation and integral sign by Fubini–Tonelli theorem, and that on the right hand side we have the variation of ϕ and the L^1 norm of f, by Lemma 2.1.6.

For any partition $\mathcal{P}$ of $[-1-\xi, 1+\xi]$ we have that

$$\sum_i |\int \hat{\phi}(x_i - y) - \hat{\phi}(x_{i+1} - y)\hat{f}(y)|dy \leq \sum_i \int |\hat{\phi}(x_i - y) - \hat{\phi}(x_{i+1} - y)||\hat{f}(y)|dy$$

exchanging the summation and integral sign and using (1) we obtain the thesis.

Suppose now $\int_{[-\xi,\xi]} \phi(x)dm(x) = 1$, by the argument above we know that the convolution integral is bounded so we can exchange the order of integration; remembering that $\hat{f}$ extends f by 0 outside $[-1, 1]$ we have then:

$$\int_{-1-\xi}^{1+\xi} \left|\int_{-\xi}^{\xi} \hat{\phi}(y)\hat{f}(x-y)dm(y)\right| dm(x) \leq \int_{-\xi}^{\xi} \hat{\phi}(y) \int_{-1-\xi}^{1+\xi} |\hat{f}(x-y)|dm(x)dm(y)$$
$$= \int_{-\xi}^{\xi} \hat{\phi}(y)||f||_{L^1([-1,1])}dm(y) = ||f||_{L^1([-1,1])}.$$

□

Remark 2.3.2 A useful characterization of bounded variation functions is the following approximation by smooth functions result, [3, Theorem 3.9]. A function $u \in L^1([a, b])$ is of bounded variation if and only if there exists a sequence u_n in $C^\infty([a, b])$ converging to u in $L^1([a, b])$ and such that

$$\lim_{n\to+\infty} \int_a^b |u_n'|dm \leq V < +\infty$$

The smallest possible constant V is the variation of u. All of the proofs about regularity in our paper can be redone by using this characterization.

2.4 Definition of the Annealed Transfer Operator

Definition 2.4.1 Let $T : [-1, 1] \to [-1, 1]$ be a measurable map. The map T induces an operator on $L : \mathcal{SM}([-1, 1]) \to \mathcal{SM}([-1, 1])$ where $\mathcal{SM}([-1, 1])$ is the space of signed measures on $[-1, 1]$, defined in the following way: if $\mu \in \mathcal{SM}([-1, 1])$ then

$$L\mu(A) = \mu(T^{-1}A)$$

for all measurable sets A. This operator is called the pushforward operator associated to T or the **transfer operator** associated to T.

The space of Lebesgue absolutely continuous measures is a vector subspace of $\mathcal{SM}([-1, 1])$; if T is non-singular with respect to Lebesgue then L preserves this subspace of absolutely continuous measures and induces an operator from $L^1([-1, 1])$ into itself called the **Perron–Frobenius** operator. We will denote by P the Perron–Frobenius operator.

Remark 2.4.2 Given an absolutely continuous probability measure $\mu = f \cdot m$, with density f, Pf is the Radon–Nikodym derivative of $L\mu$ with respect to m [20].

Remark 2.4.3 By definition, for any measurable function ϕ, the pushforward operator satisfies the following duality formula

$$\int_{-1}^{1} \phi d(L\mu) = \int_{-1}^{1} \phi \circ T d\mu.$$

The following is a collection of basic properties of the Perron–Frobenius operator P, that are proved in the first pages of [20], whose proof we omit.

Lemma 2.4.4 [20] *The following statements are true.*

(1) *Pf is the unique function in $L^1([-1, 1])$ such that for all test function in $L^\infty(m)$:*

$$\int_{-1}^{1} \phi \cdot Pf dm = \int_{-1}^{1} \phi \circ T \cdot f dm,$$

(2) *P is a positive linear operator, and $||P||_{L^1([-1,1])} = 1$,*
(3) *if f is a density, then Pf is a density.*

Definition 2.4.5 We will call **boundary condition** one of the two following maps:

- $\pi_P(x) = x \bmod 2$, called a **periodic boundary conditions**,
- $\pi_R(x) = (\min_{i\in\mathbb{Z}} |(x + 1) - 4i|) - 1$, called a **reflecting boundary conditions**.

When the choice of the boundary condition is unimportant we will denote a boundary condition by π. We will denote by π_* the push-forward map acting on measures by

$$(\pi_*\mu)(A) = \mu(\pi^{-1}(A)).$$

Remark 2.4.6 In the definition of π_P above we choose as representatives of the equivalence relation classes the points in $(-1, 1]$.

Remark 2.4.7 By abuse of notation π_* will denote also the map that π_* it induces on densities, i.e, if μ has density g, then $\pi_*(g)$ is the density of $\pi_*\mu$; refer to Lemma 2.4.10 for the conditions under which this map is well defined and their proof.

Remark 2.4.8 The map π_* is well defined only on measures μ on $\mathbb{R}$ such that there exists an interval $[a, b]$ such that

$$\mu(\mathbb{R} \setminus [a, b]) = 0;$$

by Lemma 2.2.7 this is true for all the measures $\hat{\rho} * \hat{\mu}$ in our treatment.

Remark 2.4.9 Let π^* be the map that associates to any ϕ bounded and measurable on $[-1, 1]$ its extension $\hat{\phi}$ such that

$$\hat{\phi}(x) = \phi(\pi(x)),$$

for a boundary condition π.

If μ is a measure on $\mathbb{R}$ such that there exists an interval $[a, b]$ such that

$$\mu(\mathbb{R} \setminus [a, b]) = 0;$$

we have that

$$\int \phi d\pi_*\mu = \int \pi^*(\phi) d\mu.$$

Lemma 2.4.10 *Let $\mu = f \cdot m$ be an absolutely continuous probability measure on $\mathbb{R}$, with density f such that $f \equiv 0$ in $\mathbb{R}\setminus[a, b]$. For any boundary condition π, $\pi_*(\mu)$ is an absolutely continuous probability measure on $[-1, 1]$.*

Moreover if f is of bounded variation, then $\pi_*\mu$ has a bounded variation density.

Proof Let π_i be the restriction of π to the interval $I_i = [-1+2i, 1+2i]$; by definition, π_i is one to one and affine. Let g be the density of μ and g_i its restriction to I_i, then $\pi_*\mu$ has density $\tilde{g} := \sum_i g_i(\pi_i^{-1}(x))$, where this sum is well defined since g has bounded support. Then

$$||\tilde{g}||_{L^1([-1,1])} \leq \sum_i ||g_i||_{L^1(I_i)} = ||g||_{L^1([a,b])}.$$

If g is of bounded variation, then:

$$\mathrm{Var}_{[-1,1]}(\tilde{g}) \leq \sum_i \mathrm{Var}_{I_i}(g_i) \leq \mathrm{Var}_{[a,b]}(g).$$

□

Definition 2.4.11 Let ρ a bounded variation function such that $\rho(x) \geq c > 0$ for all $x \in [-1, 1]$, $\rho(x) = 0$ outside $[-1, 1]$ and $\int_{-1}^{1} \rho(x)dm = 1$; we will call such a function a **mother noise kernel**.

In the following, define

$$\rho_\xi(x) := \frac{1}{\xi}\rho\left(\frac{x}{\xi}\right).$$

We will call ξ the **amplitude of the noise**.

Definition 2.4.12 Let $T : [-1, 1] \to [-1, 1]$ be a measurable non-singular function; a random dynamical system with noise amplitude ξ with initial condition x_0 is a sequence of random variables

$$X_0 = x_0, \quad X_{n+1} = \pi(T(X_n) + \Omega_\xi)$$

where Ω_ξ is a random variable with probability density ρ_ξ and π is either a periodic or reflecting boundary condition.

Definition 2.4.13 The annealed transfer operator L_ξ associated to the system with noise is defined by

$$L_\xi \mu = \pi_*(\hat{\rho}_\xi * \widehat{L\mu})$$

where π_* can be either periodic or reflecting boundary conditions.

Lemma 2.4.14 *The operator L_ξ induces an operator P_ξ acting on densities such that*

$$P_\xi f = \pi_*(\hat{\rho}_\xi * \widehat{Pf}).$$

Proof Let $\mu = f \cdot m$ be an absolutely continuous probability measure with density f.

By Definition 2.4.1 we have that

$$L\mu = Pf \cdot m.$$

By Lemma 2.2.7, we have that

$$\hat{\rho}_\xi * \widehat{L\mu} = (\hat{\rho}_\xi * \widehat{Pf}) \cdot m,$$

where the Lebesgue measure on the right handside is defined on $\mathbb{R}$. Remark that by 2.2.7 the support of $\hat{\rho} * \widehat{Pf}$ is contained in $[-1-\xi, 1+\xi]$.

Referring to Remark 2.4.7, we have that

$$\pi_*(\hat{\rho}_\xi * \widehat{L\mu}) = \pi_*(\hat{\rho}_\xi * \widehat{Pf}) \cdot m$$

where on the right handside m is defined on $[-1, 1]$.

Remark that by Lemmas 2.3.1 and 2.4.10, and the fact that P sends densities in densities, we have that P_ξ is a well defined operator on densities. □

Remark 2.4.15 It is worth remarking that $L_\xi \delta_y = \pi_*(\hat{\rho}_\xi(x - T(y)) \cdot m(x))$, which, by Lemma 2.2.2, Item (2) is absolutely continuous with respect to Lebesgue, with bounded variation density.

Definition 2.4.16 Let μ_ξ be a fixed point for L_ξ, i.e.,

$$L_\xi \mu_\xi = \mu_\xi.$$

We will call μ_ξ a **stationary measure** for μ_ξ.

Remark 2.4.17 If P_ξ is the Perron–Frobenius operator operating on densities, and f_ξ is a fixed point of this operator

$$P_\xi f_\xi = f_\xi$$

then $\mu_\xi = f_\xi \cdot m$, where m is the Lebesgue measure is a stationary measure.

The following theorem is a consequence of Birkhoff ergodic theorem and the skew product view of random dynamical systems, we refer to [25], and allows us to connect the notion of stationary measure and the notion of random dynamical system.

Theorem 2.4.18 (Birkhoff Ergodic Theorem) *Suppose L_ξ has a unique stationary measure μ_ξ, let $\phi \in L^1(\mu_\xi)$. Then, for μ_ξ almost every initial condition x_0 and with probability one*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(X_i) = \int \phi d\mu_\xi.$$

Remark 2.4.19 We state the ergodic theorem in this weaker form, requiring uniqueness of the stationary measure to simplify the treatment and avoid to define the notion of ergodicity for stationary measures.

Sketch of proof It is possible to associate to our random dynamical system with additive noise a skew product $F : \Omega \times [-1, 1] \to \Omega \times [-1, 1]$, where $\Omega = [-\xi, \xi]^{\mathbb{N}}$, $\sigma : \Omega \to \Omega$ is the shift map and, for $\omega \in \Omega$, $x \in [-1, 1]$ the skew product is defined as

$$F(\omega, x) = (\sigma\omega, \pi(T(x) + y)),$$

where $y = (\omega)_0$ is the first entry of ω, and π is the boundary condition.

Denote by ν the product measure induced by $\rho_\xi \cdot m$ on Ω, following the proof of [25, Proposition 5.4] verbatim, we can see that μ_ξ is stationary if and only if $\nu \times \mu_\xi$ is invariant for F;

We show the “if” claim; let $\psi(\omega, x)$ be a measurable function on $\Omega \times [-1, 1]$, and let $\phi(x) = \int \psi(\omega, x)d\nu(\omega)$; then

$$\begin{aligned}\int\int \psi(\omega, x)d\nu(\omega)d\mu_\xi(x) &= \int \phi(x)d\mu_\xi(x)\\ &= \int \phi(x)dL_\xi\mu_\xi(x) = \int\int \phi(\pi(T(x)+y))\rho_\xi(y)dm(y)d\mu_\xi(x)\\ &= \int\int\int \psi(\omega, \pi(T(x)+y))d\nu(\omega)\rho_\xi(y)dm(y)d\mu_\xi(x), \qquad (2)\end{aligned}$$

since the product measure ν is invariant for the shift and by definition

$$\nu = (\rho_\xi \cdot m) \otimes \nu,$$

we have that (2) is equal to

$$\int\int \psi(\sigma(\omega), \pi(T(x)+y))d\nu(\omega)d\mu_\xi(x) = \int\int \psi dL_F(\nu \times \mu)$$

We show the “only if” claim; let $\phi : [-1, 1] \to \mathbb{R}$ be bounded and measurable, define $\psi(\omega, x) = \phi(x)$, then, recalling Remark 2.4.9 we have

$$\begin{aligned}\int \phi(x)d(L_\xi\mu_\xi)(x) &= \int\int \phi(\pi(T(x)+y))\hat{\rho}_\xi(y)dm(y)d\mu_\xi(x)\\ &= \int\int \psi(\omega, \pi(T(x)+y))\hat{\rho}_\xi(y)dm(y)d\mu_\xi(x)\\ &= \int\int \psi(\sigma\omega, \pi(T(x)+y))\hat{\rho}_\xi(y)dm(y)d\mu_\xi(x)\\ &= \int \psi(\omega, x)dL_F(\nu \times \mu_\xi)\\ &= \int\int \psi(\omega, x)d(\nu \times \mu_\xi) = \int \phi(x)d\mu_\xi.\end{aligned}$$

By [25, Theorem 5.13] and unicity of μ_ξ we get that μ_ξ is an ergodic stationary measure (we refer to [25] Sect. 5.3 for a definition), therefore $\nu \times \mu_\xi$ is an ergodic invariant measure for F and the statement follows. □

2.5 Regularization Properties of the Annealed Transfer Operator

The ergodic theorem tells us that if we want to understand the statistical properties of a random dynamical system, we need to prove uniqueness of its stationary measure and study its properties. Our plan is to show that under some assumptions the random dynamical system admits a unique stationary measure, with density of bounded variation.

Corollary 2.5.1 *The operator P_ξ is a bounded operator from L^1 to BV, such that*

$$Var_{[-1,1]}(P_\xi f) \leq \left(Var_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right)||f||_{L^1([-1,1])},$$

which in turn implies that $Var_{[-1,1]}(P_\xi f) \leq 3||\rho_\xi||_{BV([-\xi,\xi])}||f||_{L^1([-1,1])}$.

Proof This follows from Lemma 2.3.1 and the proof of Lemma 2.4.10, i.e.,

$$\begin{aligned}\mathrm{Var}_{[-1,1]}(P_\xi f) &= \mathrm{Var}_{[-1,1]}(\pi_*(\hat{\rho}_\xi * \widehat{Pf})) \\ &\leq \mathrm{Var}_{[-1-\xi,-1+\xi]}(\hat{\rho}_\xi * \widehat{Pf}) \\ &\leq \left(\mathrm{Var}_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right)||Pf||_{L^1([-1,1])}\end{aligned}$$

and the fact that $||P||_{L^1\to L^1} \leq 1$. As in many other occasions, we use that $||\rho_\xi||_{BV([-\xi,\xi])} \geq \sup_{[-\xi,\xi]}|\rho_\xi(x)|$ to give the following bound

$$\left(\mathrm{Var}_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right) \leq 3||\rho_\xi||_{BV([-\xi,\xi])}.$$

□

Remark 2.5.2 In particular, if f is a density (Definition 2.1.2), we have that

$$\mathrm{Var}_{[-1,1]}(P_\xi f) \leq \mathrm{Var}_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|.$$

Corollary 2.5.3 (Big noise amplitude limit) *Let f_ξ be a density which is a fixed point of P_ξ; then*

$$\begin{aligned}Var_{[-1,1]}(f_\xi) &\leq \left(Var_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right)||f_\xi||_{L^1([-1,1])} \\ &= \left(Var_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right).\end{aligned}$$

Moreover this implies that

$$\lim_{\xi\to+\infty} Var_{[-1,1]}(f_\xi) = 0,$$

and therefore

$$\lim_{\xi\to+\infty} ||f_\xi - \frac{1}{2}||_{BV([-1,1])} = 0.$$

Proof This follows from Corollary 2.5.1:

$$\mathrm{Var}_{[-1,1]}(f_\xi) = \mathrm{Var}_{[-1,1]}(P_\xi f_\xi) \leq \left(\mathrm{Var}_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right).$$

The second statement follows from

$$\begin{aligned}\mathrm{Var}_{[-1,1]}(f_\xi) &\leq \left(\mathrm{Var}_{[-\xi,\xi]}(\rho_\xi) + 2\sup_{[-\xi,\xi]}|\rho_\xi(x)|\right) \\ &= \frac{1}{\xi}\left(\mathrm{Var}_{[-1,1]}(\rho) + 2\sup_{[-1,1]}|\rho(x)|\right),\end{aligned}$$

so

$$\lim_{\xi\to+\infty} \mathrm{Var}_{[-1,1]}(f_\xi) = 0,$$

which implies the thesis. □

Remark 2.5.4 Remark that Corollary 2.5.1 and 2.5.3 do not depend on our choice of boundary condition.

Remark 2.5.5 Corollary 2.5.3 tells us that for any bounded variation noise kernel, as the amplitude of the noise increases, the orbits of the random dynamical system distribute themselves uniformly in the interval $[-1, 1]$.

Definition 2.5.6 Let

$$\mathcal{U}_0 = \{f \in L^1([-1,1]) \mid \int f\,dm = 0\}.$$

We call $\mathcal{U}_0$ the vector subspace of average 0 measures; by abuse of notation we denote by $\mathcal{U}_0$ also its intersection with $BV([-1,1])$. We say P_ξ **contracts the space of average zero functions in** L^1 if

$$||P_\xi^n|_{\mathcal{U}_0}||_{L^1\to L^1} \leq C\theta^n$$

for constants $C > 0$, $0 < \theta < 1$. We say P_ξ **contracts the space of average zero functions in** BV if

$$||P_\xi^n|_{\mathcal{U}_0}||_{BV\to BV} \leq \tilde{C}\tilde{\theta}^n$$

for constants $\tilde{C} > 0$, $0 < \tilde{\theta} < 1$.

Lemma 2.5.7 *The operator P_ξ contracts the space of average zero functions in L^1 if and only if it contracts the space of average of average zero functions in BV.*

Proof By Lemma 2.3.1, we have that

$$||P_\xi||_{L^1\to BV} \leq 3||\rho_\xi||_{BV([-\xi,\xi])}.$$

Suppose P_ξ contracts the space of average 0 functions in BV. Let f be an average 0 function in L^1, then

$$||P_\xi^n f||_{L^1} \leq ||P_\xi^n f||_{BV} \leq ||P_\xi^{n-1}|_{\mathcal{U}_0}||_{BV\to BV} 3||\rho_\xi||_{BV([-\xi,\xi])}||f||_{L^1},$$

which implies that

$$||P_\xi^n|_{\mathcal{U}_0}||_{L^1} \leq \frac{3\tilde{C}||\rho_\xi||_{BV([-\xi,\xi])}}{\tilde{\theta}}\tilde{\theta}^n,$$

i.e., P_ξ contracts the space of average 0 functions in L^1. If P_ξ contracts the space of average 0 functions in L^1 we have that, if f is an average 0 function in BV

$$||P_\xi^n f||_{BV} \leq 3||\rho_\xi||_{BV([-\xi,\xi])}||P_\xi^{n-1} f||_{L^1\to L^1} \leq 3||\rho_\xi||_{BV([-\xi,\xi])}\cdot C\theta^{n-1}||f||_{L^1};$$

since $||f||_{L^1} \leq ||f||_{BV}$ this implies that

$$||P_\xi^n|_{\mathcal{U}_0}||_{BV} \leq \frac{3||\rho_\xi||_{BV([-\xi,\xi])}\cdot C}{\theta}\theta^n,$$

i.e., that P_ξ contracts the space of average functions in BV. □

Lemma 2.5.8 *If P_ξ contracts the space of average 0 functions in L^1 (or equivalently in BV), then L_ξ has a unique stationary measure.*

Proof We prove by contradiction that the stationary measure is unique: let μ and ν be stationary measures. Since $L_\xi \mu = \mu$ and $L_\xi \nu = \nu$ we have that μ and ν are absolutely continuous with respect to Lebesgue, with densities f and g respectively. Now, $P_\xi f = f$ and $P_\xi g = g$, and, since P_ξ contracts the space of average 0 measures, we have that for any n

$$||f - g||_{L^1} = ||P_\xi^n (f - g)||_{L^1} \leq C\theta^n ||f - g||_{L^1}.$$

Take N such that $C\theta^N < 1$, the inequality above then implies that $||f - g||_{L^1} = 0$, which in turn implies that $\mu = \nu$. □

We will now generalize of a result in [11]: if for some noise amplitude the operator contracts the space of average 0 functions in L^1, then for all bigger amplitudes the annealed operators also contracts the space of average 0 functions in L^1. We start by an auxiliary Lemma and Corollary.

Lemma 2.5.9 *Let ρ be a mother noise kernel, ρ_ξ its rescaling, μ be a probability measure on* $[-1, 1]$, *$\hat{\mu}$ its extension to $\mathbb{R}$ by $\hat{\mu}(A) = \mu(A \cap [-1, 1])$, then, for any measurable subset A and for each $\hat{\xi} > \xi$ we have that*

$$\pi_*(\hat{\rho}_{\hat{\xi}} * \hat{\mu})(A) \geq \frac{c}{||\rho||_{BV}} \frac{\xi}{\hat{\xi}} \pi_*(\hat{\rho}_\xi * \hat{\mu})(A).$$

Proof We remember that $\rho_\xi(x) = \frac{1}{\xi}\rho(x/\xi)$ and that $\rho(x) \geq c > 0$ for all $x \in [-1, 1]$ by Definition of 2.4.11. Therefore

$$\rho_\xi(x) \geq \frac{c}{\xi}$$

for all $x \in [-\xi, \xi]$.

We have that

$$\pi_*(\hat{\rho}_{\hat{\xi}} * \hat{\mu})(A) = \int_{[-\hat{\xi},\hat{\xi}]} \hat{\rho}_{\hat{\xi}}(y)\hat{\mu}(A - y)dm(y);$$

by the observation above we have that

$$\int_{[-\hat{\xi},\hat{\xi}]} \hat{\rho}_{\hat{\xi}}(y)\hat{\mu}(A - y)dm(y) \geq \frac{c}{\hat{\xi}} \int_{[-\hat{\xi},\hat{\xi}]} \hat{\mu}(A - y)dm(y).$$

Now, since

$$||\rho_\xi||_\infty \leq ||\rho_\xi||_{BV} \leq \frac{||\rho||_{BV}}{\xi}$$

we have that

$$\pi_*(\hat{\rho}_\xi * \hat{\mu})(A) \leq \frac{||\rho||_{BV}}{\xi} \int_{[-\xi,\xi]} \hat{\mu}(A - y)dm(y).$$

Since $\hat{\xi} > \xi$ and $\hat{\mu}(A - y)$ is nonnegative for all y we have

$$\int_{[-\hat{\xi},\hat{\xi}]} \hat{\rho}_{\hat{\xi}}(y)\hat{\mu}(A - y)dm(y) \geq \frac{c}{\hat{\xi}} \int_{[-\hat{\xi},\hat{\xi}]} \hat{\mu}(A - y)dm(y) \geq$$
$$\frac{c}{\hat{\xi}} \int_{[-\xi,\xi]} \hat{\mu}(A - y)dm(y) \geq \frac{c}{||\rho||_{BV}} \frac{\xi}{\hat{\xi}} \int_{[-\xi,\xi]} \hat{\rho}_\xi(y)\hat{\mu}(A - y)dm(y).$$

and the thesis follows. □

Corollary 2.5.10 *Let ν be a probability measure in $[-1, 1]$. Then, letting $\tau = \frac{c}{||\rho||_{BV}} \frac{\xi}{\hat{\xi}}$, we have that*

$$L_{\hat{\xi}}\mu(A) \geq \tau L_{\xi}\mu(A),$$

for all measurable subset A.

Proof Use Lemma 2.5.9 with $\mu = L\nu$. □

We can now prove that mixing for some noise amplitude implies mixing for all bigger noise amplitudes.

Lemma 2.5.11 *Suppose P_{ξ} contracts the space of average 0 functions in L^1 for $\xi > 0$; then $P_{\hat{\xi}}$ contracts the space of average 0 functions for any $\hat{\xi} > \xi$.*

Proof By Lemma 2.2.2, Item (2), we know that $L_{\xi}\delta_x$ is an absolutely continuous probability measure and by Corollary 2.5.10 we have that for any measurable subset A, and any $x \in [-1, 1]$:

$$(L^m_{\hat{\xi}}\delta_x)(A) \geq \tau^m (L^m_{\xi}\delta_x)(A).$$

By hypothesis, P_{ξ} is contracting the space of average 0 functions in L^1, so that, for any $x \in [-1, 1]$, if f_{ξ} is the density of the stationary measure and g is the density of $L^m_{\xi}\delta_x$, we have that

$$||f_{\xi} - P^{m-1}_{\xi} g||_{L^1} = ||P^{m-1}_{\xi}(f_{\xi} - g)||_{L^1} \leq 2C\theta^{m-1},$$

which in turn implies that for any measurable subset A

$$|(L^m_{\xi}\delta_x)(A) - \mu_{\xi}(A)| \leq 2C\theta^{m-1}.$$

Let N such that $2C\theta^{N-1} < 1$, and let

$$\nu_N = \tau^N(1 - 2C\theta^{N-1})\mu_{\xi},$$

then

$$(L^N_{\hat{\xi}}\delta_x)(A) \geq \nu_N(A)$$

for all x and for all measurable A.

Remember that, if μ and ν are absolutely continuous measures with respect to Lebesgue, with densities f and g respectively, we have that the total variation norm for measures (we refer to [18] for its definition) is related by the L^1 norm by the following equation:

$$||\mu - \nu||_{TV} = \frac{1}{2}||f - g||_{L^1}.$$

Then, by [18, Theorem 16.2.4] and the fact that L_{ξ} maps measure into absolutely continuous measures, i.e., item 2 in Lemma 2.2.7, we have that

$$||P^n_{\hat{\xi}}|_{\mathcal{U}_0}||_{L^1} \leq 4\rho^{\lfloor n/N \rfloor};$$

where $\rho = 1 - \tau^N(1 - 2C\theta^{N-1})$. □

Lemma 2.5.12 *There exists $C > 0$, $0 < \theta < 1$ such that for all $\xi \geq 1$*

$$||P^n_{\xi}|_{\mathcal{U}_0}||_{L^1} \leq C\theta^n.$$

Proof Recall that $\rho_\xi(x) = \frac{1}{\xi}\rho(x/\xi)$. Therefore, for ξ in $[k, k+1)$, where k is a positive natural number, we have that

$$\rho_\xi(x) \geq \frac{c}{k+1},$$

and we have that for all $x \in [-1, 1]$, due to boundary conditions,

$$L_\xi \delta_x(A) \geq \frac{k}{k+1} c \cdot m(A).$$

This implies that for all $\xi \geq 1$, we have that

$$L_\xi \delta_x(A) \geq \frac{c}{2} \cdot m(A).$$

By [18, Theorem 16.2.4] and the fact that L_ξ maps measure into absolutely continuous measures, i.e., Lemma 2.2.7, we have that

$$||P_\xi^n|_{\mathcal{U}_0}||_{L^1} \leq 4\rho^n;$$

where $\rho = 1 - c/2$. □

Corollary 2.5.13 *Suppose P_ξ contracts the space of average 0 functions in L^1 for $\xi > 0$; then there exists $C > 0$, $0 < \theta < 1$ such that*

$$||P_{\hat{\xi}}^n|_{\mathcal{U}_0}||_{L^1} \leq C\theta^n$$

for all $\hat{\xi} > \xi$.

Proof Using a compactness argument, for all $\hat{\xi}$ in $(\xi, 1]$ we have a uniform bound from Lemma 2.5.11; for $\xi > 1$ we have a uniform bound from Lemma 2.5.12. □

2.6 L^r Continuity of the Stationary Measure with Respect to the Noise Size

In this subsection we prove continuity in $L^r([-1, 1])$ of the stationary measure with respect to the noise size at a fixed noise size $\xi > 0$.

Lemma 2.6.1 *Suppose that P_ξ contracts the space of average 0 functions in L^1. Moreover,suppose that there exists a $0 < \epsilon < \xi$ such that for all $\hat{\xi}$ in $(\xi - \epsilon, \xi + \epsilon)$ the operator $P_{\hat{\xi}}$ has a unique fixed density $f_{\hat{\xi}}$.*

Then

$$\lim_{\hat{\xi} \to \xi} ||f_{\hat{\xi}} - f_\xi||_{L^1} = 0.$$

Proof A bounded variation function on the interval has a countable set of discontinuity points, since it can be written as the difference of two monotone functions [3].

Moreover, a bounded variation function is bounded; fix ξ and let Ω_ξ be the set of discontinuities of ρ_ξ. We claim that

$$\lim_{\hat{\xi} \to \xi} ||\hat{\rho}_{\hat{\xi}} - \hat{\rho}_\xi||_{L^1} = 0,$$

where $\hat{\rho}_{\hat{\xi}}$, $\hat{\rho}_\xi$ are the extensions of $\rho_{\hat{\xi}}$ and ρ_ξ respectively to $\mathbb{R}$.

Observe that

$$||\hat{\rho}_{\hat{\xi}} - \hat{\rho}_{\xi}||_{L^1} = \int_{[-\hat{\xi},\hat{\xi}]\setminus\Omega_\xi} |\hat{\rho}_{\hat{\xi}}(x) - \hat{\rho}_{\xi}|dm \le \max\left(\frac{1}{\hat{\xi}}, \frac{1}{\xi}\right)||\rho||_{BV}.$$

By the dominated convergence theorem we have then that

$$\lim_{\hat{\xi}\to\xi} \int_{[-\hat{\xi},\hat{\xi}]\setminus\Omega_\xi} |\hat{\rho}_{\hat{\xi}}(x) - \hat{\rho}_{\xi}(x)|dm = \int_{[-\hat{\xi},\xi)\cup(\xi,\hat{\xi}]} \lim_{\hat{\xi}\to\xi} |\hat{\rho}_{\hat{\xi}}(x)|dm + \int_{[-\xi,\xi]\setminus\Omega_\xi} \lim_{\hat{\xi}\to\xi} |\hat{\rho}_{\hat{\xi}}(x) - \hat{\rho}_{\xi}(x)|dm$$

which goes to 0 as $\hat{\xi}$ goes to ξ.

By L^1 continuity of the convolution this in turn implies that

$$\lim_{\hat{\xi}\to\xi} ||P_{\hat{\xi}} - P_\xi||_{L^1\to L^1} = 0,$$

and

$$||f_\xi - f_{\hat{\xi}}||_{L^1} \le ||P_\xi^N(f_\xi - f_{\hat{\xi}})||_{L^1} + ||P_\xi^N f_{\hat{\xi}} - P_{\hat{\xi}}^N f_{\hat{\xi}}||_{L^1}.$$

Since $||P_\xi^i|_{\mathcal{U}_0}||_{L^1} < C\theta^i$ where $0 < \theta < 1$, there exists a positive N such that $C\theta^N < 1/2$, since $f_\xi - f_{\hat{\xi}}$ is an average 0 function in L^1 we have that

$$||f_\xi - f_{\hat{\xi}}||_{L^1} \le \frac{1}{2}||f_\xi - f_{\hat{\xi}}||_{L^1} + ||P_\xi^N f_{\hat{\xi}} - P_{\hat{\xi}}^N f_{\hat{\xi}}||_{L^1},$$

and we estimate the right hand side by telescopizing the difference of powers:

$$\begin{aligned}
||P_\xi^N f_{\hat{\xi}} - P_{\hat{\xi}}^N f_{\hat{\xi}}||_{L^1} &\le \sum_{k=0}^{N-1} ||P_\xi^k|_{\mathcal{U}_0}||_{L^1\to L^1}||P_\xi - P_{\hat{\xi}}||_{L^1\to L^1}||P_{\hat{\xi}}^{N-k-1} f_{\hat{\xi}}||_{L^1} \\
&\le \sum_{k=0}^{N-1} ||P_\xi^k|_{\mathcal{U}_0}||_{L^1\to L^1}||P_\xi - P_{\hat{\xi}}||_{L^1\to L^1}||f_{\hat{\xi}}||_{L^1} \\
&\le \frac{C}{1-\theta}||P_\xi - P_{\hat{\xi}}||_{L^1\to L^1};
\end{aligned}$$

where we used that $||f_{\hat{\xi}}||_{L^1} = 1$. This implies that

$$||f_\xi - f_{\hat{\xi}}||_{L^1} \le 2C\frac{1}{1-\theta}||P_\xi - P_{\hat{\xi}}||_{L^1\to L^1}$$

Taking the limit as $\hat{\xi} \to \xi$ we conclude the proof. □

Remark 2.6.2 The same argument can be used to prove right continuity of the stationary measure in L^1. The main difference is that we can drop the hypothesis of the uniqueness of the stationary measure since by Corollary 2.5.13 the contraction of the space of average 0 functions at ξ implies uniform contraction for all $\hat{\xi} \ge \xi$, so the uniqueness of $f_{\hat{\xi}}$ follows.

Remark 2.6.3 Lemma 2.6.1 proves the continuity at ξ in L^1 norm of the stationary density if the operator P_ξ is contracting the space of average 0 measures and the stationary measure is unique in a neighborhood of ξ.

Even if we have uniform contraction rates, we can only prove continuity in L^1 of the stationary density as a function of ξ.

More regular noise kernel allow us to prove stronger regularity of f_ξ as a function of ξ, which reflects in stronger regularity of the Birkhoff averages of observables as a function of ξ.

Corollary 2.6.4 *Suppose that P_ξ contracts the space of average 0 functions in L^1. Moreover,suppose that there exists a $0 < \epsilon < \xi$ such that for all $\hat{\xi}$ in $(\xi - \epsilon, \xi + \epsilon)$ the operator $P_{\hat{\xi}}$ has a unique fixed density $f_{\hat{\xi}}$.*

Then, for any $1 < r < +\infty$

$$\lim_{\hat{\xi}\to\xi} ||f_{\hat{\xi}} - f_\xi||_{L^r} = 0.$$

Proof Recall that if f_ξ is a fixed point of P_ξ with $||f_\xi||_{L^1} = 1$ we have that

$$||f_\xi||_{BV} = ||P_\xi f_\xi||_{BV} \leq ||P_\xi||_{L^1\to BV}||f_\xi||_{L^1} \leq 3||\rho_\xi||_{BV}.$$

The BV norm bounds from above the L^∞-norm, so

$$||f_\xi - f_{\hat{\xi}}||_\infty \leq 6\frac{||\rho||_{BV}}{\xi - \epsilon}.$$

Therefore, $f_\xi - f_{\hat{\xi}}$ belongs to $L^1 \cap L^\infty$ and by the classical L^p interpolation inequality we have that

$$||f_\xi - f_{\hat{\xi}}||_{L^r} \leq \left(||f_\xi - f_{\hat{\xi}}||_{L^1}\right)^{1/r} \cdot \left(||f_\xi - f_{\hat{\xi}}||_{L^\infty}\right)^{1-1/r}.$$

Therefore,

$$\lim_{\hat{\xi}\to\xi} ||f_\xi - f_{\hat{\xi}}||_{L^r} \leq \left(6\frac{||\rho||_{BV}}{\xi - \epsilon}\right)^{1-1/r} \lim_{\hat{\xi}\to\xi}\left(||f_\xi - f_{\hat{\xi}}||_{L^1}\right)^{1/r} = 0.$$

□

Corollary 2.6.5 *Let $\phi \in L^p([-1, 1])$, with $p > 1$; suppose there exists a ξ_0 such that P_{ξ_0} contracts the space of average 0 functions in BV or equivalently L^1. Then, the function*

$$A_\phi(\xi) = \int_{-1}^{1} \phi d\mu_\xi$$

is well defined and continuous for all $\xi \geq \xi_0$.

Proof By Lemma 2.5.11 we have that P_ξ contracts the space of average 0 functions in BV for all $\xi \geq \xi_0$ This implies by Lemma 2.5.8 that for each $\xi \geq \xi_0$ there exists a unique stationary measure μ_ξ with density f_ξ in BV. By Lemma 2.6.4, fixing $\epsilon < \xi - \xi_0$ and letting $q > 0$ be such that $1/q + 1/p = 1$ we have that, for $\hat{\xi} \in (\xi - \epsilon, \xi + \epsilon)$

$$\lim_{\hat{\xi}\to\xi} |\int_{-1}^{1} \phi f_\xi dm - \int_{-1}^{1} \phi f_{\hat{\xi}} dm| \leq \lim_{\hat{\xi}\to\xi} ||\phi||_{L^p}||f_\xi - f_{\hat{\xi}}||_{L^q} = 0$$

which implies the continuity of A_ϕ for all $\xi > \xi_0$. The proof of 2.6.1 can be redone verbatim for right continuity at ξ_0 as explained in remark 2.6.2, which implies right continuity at ξ_0 and the thesis. □

Corollary 2.6.6 *Let $\phi \in L^1([-1, 1])$; suppose there exists an $0 < \xi_0 < +\infty$ such that P_{ξ_0} contracts the space of average 0 functions in BV or equivalently L^1. Then, if*

$$A_\phi(\xi) = \int_{-1}^{1} \phi d\mu_\xi$$

we have that

$$\lim_{\xi \to +\infty} A_\phi(\xi) = \int_{-1}^{1} \phi \frac{1}{2} dm.$$

Proof As in Corollary 2.6.5 the function is well defined for all $\xi \geq \xi_0$. We have that

$$\lim_{\xi \to +\infty} |\int_{-1}^{1} \phi f_\xi dm - \int_{-1}^{1} \phi \frac{1}{2} dm| \leq \lim_{\xi \to +\infty} ||\phi||_{L^1} ||f_\xi - \frac{1}{2}||_{L^\infty} = 0,$$

recalling that the BV norm bounds from above the L^∞ norm, the thesis follows from Corollary 2.5.3 . □

2.7 Continuity with Respect to the Base Dynamic *T*

To study the behavior as the base dynamic varies, we will use the following arguments by M. Monge, that was proved for a version of [11].

Definition 2.7.1 A piecewise continuous map T on $[-1, 1]$ is a function $T : [-1, 1] \to [-1, 1]$ such that there is partition $\{I_i\}_{1 \leq i \leq k}$ of $[-1, 1]$ made of intervals I_i such that T has a continuous extension to the closure $\bar{I}_i$ of each interval. We call this partition the **continuity partition** of T.

If two piecewise continuous maps T_1 and T_2 share the same continuity partition we define

$$||T_1 - T_2||_\infty = \max_i \sup_{x \in I_i} |T_1(x) - T_2(x)|.$$

Remark 2.7.2 Remark that a piecewise continuous map is uniformly continuous when restricted to each I_i in its continuity partition.

Remark 2.7.3 The condition that two maps share the same continuity partition is used to generalize the sup distance on continuous maps to piecewise continuous maps; as observed by one of the referees the arguments in the rest of the section do not depend strictly on it but the treatment is easier if we assume it.

Definition 2.7.4 The Wasserstein–Kantorovich distance of two probability measures is defined as

$$W(\mu, \nu) = \sup_{\mathrm{Lip}(\phi) \leq 1, ||\phi||_\infty = 1} \left| \int \phi d\mu - \int \phi d\nu \right|$$

Remark 2.7.5 We refer to [11] for the properties of the Wasserstein–Kantorovich distance we use. It is worth observing that

$$W(\delta_p, \delta_q) = |p - q|.$$

We now give a proof of [11, Lemma 51], starting by proving another property of bounded variation functions.

Lemma 2.7.6 *Let ϕ be a bounded variation function on $[a, b]$, zero outside of $[a, b]$. Let τ_h be the translation operator $\tau_h(\phi)(x) = \phi(x + h)$. Then*

$$||\tau_h\phi - \phi||_{L^1([a-h,b+h])} \leq h\left(Var_{[a,b]}(\phi) + 4\sup_{[a,b]}|\phi(x)|\right).$$

Proof Without loss of generality, suppose $h > 0$, the negative case is analogous.

We start by observing that

$$\begin{aligned}&||\tau_h(\phi) - \phi||_{L^1([a-h,b+h])}\\&= \int_{a-h}^{a}|\phi(x+h)|dx + \int_{a}^{b-h}|\phi(x+h) - \phi(x)|dx + \int_{b-h}^{b}|\phi(x)|dx\\&\leq \int_{a}^{b-h}|\phi(x+h) - \phi(x)|dx + 2\sup_{[a,b]}|\phi(x)|\cdot h.\end{aligned}$$

Let now N be the biggest integer such that $Nh \leq b - h - a$; then, the intervals $I_i = [a + ih, a + (i + 1)h]$ for $i = 0, \dots, N - 1$, and $J = [a + Nh, b - h]$ are a partition of $[a, b - h]$; remark that $m(J) < h$. Then

$$\begin{aligned}\int_{a}^{b-h}|\phi(x+h) - \phi(x)|dx &= \sum_i\int_{I_i}|\phi(x+h) - \phi(x)|dx + \int_J|\phi(x+h) - \phi(x)|dx\\&\leq \sum_{i=0}^{N-1}\int_0^h|\phi(a+(i+1)h+z) - \phi(a+ih+z)|dz + 2\sup_{[a,b]}|\phi(x)|\cdot h,\end{aligned}$$

where on each U_i we used the change of coordinates $x = a + ih + z$. Now, we have that

$$\int_0^h\sum_i|\phi(a+(i+1)h+x) - \phi(a+ih+x)|dx \leq \int_0^h \text{Var}_{[a+x,b+x]}(\tau_x\phi)dx.$$

since the variation is translation invariant, we have then that

$$\int_a^{b-h}|\phi(x+h) - \phi(x)|dx \leq \left(\text{Var}_{[a,b]}(\phi) + 2\sup_{[a,b]}|\phi(x)|\right)\cdot h$$

Summarizing, using the fact that ϕ is zero outside of $[a, b]$, we have

$$||\tau_h(\phi) - \phi||_{L^1([a-h,b+h])} \leq \left(\text{Var}_{[a,b]}(\phi) + 4\sup_{[a,b]}|\phi(x)|\right)h.$$

□

Lemma 2.7.7 *Let ϕ be a bounded variation function on $[a, b]$, zero outside of $[a, b]$, and let $\psi \in L^\infty(\mathbb{R})$. Then, their convolution $\phi * \psi$ is a Lipschitz function with Lipschitz constant bounded above by $\left(Var_{[a,b]}(\phi) + 4\sup_{[a,b]}|\phi(x)|\right)||\psi||_\infty$.*

Proof This follows from the definition of convolution

$$|\phi * \psi(x+h) - \phi * \psi(x)| = \left| \int_{\mathbb{R}} (\phi(x+h-y) - \phi(x-y))\, \psi(y) dy \right|$$

$$\leq ||\psi||_{L^\infty} \int_{\mathbb{R}} |\phi(x+h-y) - \phi(x-y)|\, dy = ||\psi||_{L^\infty} ||\tau_h(\tau_x \phi) - \tau_x \phi||_{L^1(\mathbb{R})}$$

$$= ||\psi||_{L^\infty} ||\tau_h(\phi) - \phi||_{L^1([a-h, b+h])} \leq ||\psi||_{L^\infty} \left(\mathrm{Var}_{[a,b]}(\phi) + 4 \sup_{[a,b]} |\phi(x)| \right) h.$$

where we used the fact that ϕ is 0 outside $[a, b]$ and invariance of the L^1 norm by τ_x. □

Lemma 2.7.8 *Let now μ and ν be probability measures on $[-1, 1]$; as in Lemma 2.2.7 we have that $\hat{\rho}_\xi * \hat{\mu}$ and $\hat{\rho}_\xi * \hat{\nu}$ are absolutely continuous with respect to Lebesgue, let f and g be their densities. Then*

$$||f - g||_{L^1([-1-\xi, 1+\xi])} \leq \left(Var(\rho_\xi) + 4 \sup_{[-\xi,\xi]} |\rho_\xi(x)| \right) W(\mu, \nu) \leq 5 ||\rho_\xi||_{BV} W(\mu, \nu).$$

Proof Recall that f and g are 0 outside $[-1-\xi, 1+\xi]$; for each $\psi \in L^\infty(\mathbb{R})$ we have that

$$\left| \int \psi(f-g) dx \right| = \left| \int \psi d(\hat{\rho}_\xi * \hat{\mu}) - \int \psi d(\hat{\rho}_\xi * \hat{\nu}) \right|$$

$$= \left| \int \int \psi(x) \hat{\rho}_\xi(x-y) dm(x) d\hat{\mu}(y) - \int \int \psi(x) \hat{\rho}_\xi(x-y) dm(x) d\hat{\nu}(y) \right|$$

$$= \left| \int (\psi * \hat{\rho}_\xi) d\hat{\mu} - \int (\psi * \hat{\rho}_\xi) d\hat{\nu} \right|.$$

By Lemma 2.7.7 and definition of Wasserstein distance we have then

$$\left| \int \psi(f-g) dx \right| \leq \left(\mathrm{Var}_{[-\xi,\xi]}(\rho_\xi) + 4 \sup_{[-\xi,\xi]} |\rho_\xi(x)| \right) ||\psi||_{L^\infty} \cdot W(\mu, \nu),$$

which in turn implies the thesis by taking as ψ the function with value 1 if $f(x) \geq g(x)$ and value -1 if $f(x) < g(x)$. □

Remark 2.7.9 The constants in the preceding lemmas are not optimal, but are enough for our goal of studying the continuity of the stationary density with respect to the parameters of the system in the presence of positive amplitude noise.

Lemma 2.7.10 *Let T_1 and $T_2 : [-1, 1] \to [-1, 1]$ be piecewise continuous nonsingular maps that share the same continuity partition and let L_{T_1}, L_{T_2} the associated transfer operators, let $f \in L^1$. Then:*

$$W(L_{T_1}(f dm), L_{T_2}(f dm)) \leq ||T_1 - T_2||_\infty ||f||_1,$$

or equivalently

$$W((P_{T_1} f) dm, (P_{T_2} f) dm) \leq ||T_1 - T_2||_\infty ||f||_1.$$

Proof Let $[a, b]$ be an interval and let $\mathcal{P} = \{p_0 = a, \ldots, p_n = b\}$ be the endpoints of a partition such that $p_{i+1} - p_i \leq D$ (in the following we will call D the diameter of the partition).

Let $f \in L^1([a,b])$ be a positive function; the projection of $f dm$ associated to the partition $\mathcal{P}$ is

$$\pi_{\mathcal{P}} f = \sum_{i=0}^{n-1} \left(\int_{p_i}^{p_{i+1}} f dm \right) \cdot \delta_{p_i}$$

where δ_{p_i} is the Dirac δ at p_i.

Then, for any Lipschitz function ϕ on $[a,b]$ we have

$$\left| \sum_{i=0}^{n-1} \int_{p_i}^{p_{i+1}} \phi(p_i) f(x) dm - \int_{p_i}^{p_{i+1}} \phi(x) f(x) dm \right| \leq \text{Lip}(\phi) ||f||_{L^1([a,b])} \cdot D,$$

which implies

$$W(\pi_{\mathcal{P}} f, f dm) \leq D ||f||_{L^1([a,b])}.$$

Fix $\epsilon > 0$, by uniform continuity there exists a D such that the image of a partition of diameter D has diameter at most ϵ; let f be a density, and let $\mathcal{P}$ be a partition of diameter D, as above.

For a Dirac δ_p at p, we have that

$$L_{T_i} \delta_p = \delta_{T_i(p)}$$

for $i = 1, 2$, which implies

$$W(L_{T_1} \delta_p, L_{T_2} \delta_p) \leq ||T_1 - T_2||_\infty$$

By triangle inequality, this implies that

$$W(L_{T_1} \pi_{\mathcal{P}} f, L_{T_2} \pi_{\mathcal{P}} f) \leq \sum_i \left(\int_{p_i}^{p_{i+1}} f dm \right) \cdot W(L_{T_1} \delta_{p_i}, L_{T_2} \delta_{p_i}) \leq ||T_1 - T_2||_\infty ||f||_{L^1}.$$

Now, any Lipschitz function ϕ is bounded; by the duality properties of the transfer operator and the Koopman operator, we have

$$\left| \int \phi L_{T_1}(f dm) - \sum_i \phi(T_1(p_i)) \int_{p_i}^{p_{i+1}} f dm \right| = \left| \sum_i \int_{p_i}^{p_{i+1}} (\phi \circ T_1(x) - \phi \circ T_1(p_i)) f dm \right|,$$

which in turn implies

$$W(L_{T_1}(f dm), L_{T_1} \pi_{\mathcal{P}} f) \leq \epsilon,$$

and similarly for T_2.

This implies that

$$W(L_{T_1}(f dm), L_{T_2}(f dm)) \leq W(L_{T_1} \pi_{\mathcal{P}} f, L_{T_2} \pi_{\mathcal{P}} f) + 2\epsilon,$$

as ϵ is arbitrary, we obtain the thesis. □

Definition 2.7.11 Let T_1 and $T_2 : [-1,1] \to [-1,1]$ be piecewise continuous nonsingular maps that share the same continuity partition. We will denote by

$$L_{\xi, T_i} = \pi_*(\rho_\xi * L_{T_i}), \quad \text{for } i = 1, 2,$$

and the associated annealed Perron–Frobenius operators

$$P_{\xi, T_i} f = \pi_*(\rho_\xi * P_{T_i}(f)) \quad \text{for } i = 1, 2.$$

Lemma 2.7.12 *Let T_1 and $T_2 : [-1,1] \to [-1,1]$ be piecewise continuous nonsingular maps that share the same continuity partition. Then for any $f \in L^1$:*

$$||P_{\xi,T_1}(f) - P_{\xi,T_1}(f)||_1 \leq ||T_1 - T_2||_\infty 5||\rho_\xi||_{BV}||f||_1$$

Proof

$$\begin{aligned}||P_{\xi,T_1}(f) - P_{\xi,T_2}(f)||_1 &= ||\pi_*||_{BV\to BV}||\hat{\rho}_\xi * (P_{T_1}(f) - P_{T_2}(f))||_1\\ &\leq 5||\rho_\xi||_{BV} \cdot W(P_{T_1}(f)dm, P_{T_2}(f)dm),\end{aligned}$$

where the operator norm of $\pi_* : BV([-1-\xi, 1+\xi]) \to BV([-1,1])$ is bounded by Lemma 2.4.10. The statement then follows by Lemma 2.7.10. □

Lemma 2.7.13 *Let T_1 and T_2 be piecewise continuous nonsingular maps that share the same continuity partition. Suppose P_{ξ,T_1} contracts the space of average 0 functions in L^1 with constants $C > 0$ and $0 < \theta < 1$ then*

$$||P^n_{\xi,T_1} f - P^n_{\xi,T_2} f||_{L^1} \leq \frac{C}{1-\theta}||T_1 - T_2||_\infty 5||\rho_\xi||_{BV}.$$

Proof This follows from a telescopization argument:

$$||P^n_{\xi,T_1} f - P^n_{\xi,T_2} f||_{L^1} \leq \sum_{i=0}^{n} ||P^i_{\xi,T_1}||_{L^1\to L^1}||P_{\xi,T_1} - P_{\xi,T_2}||_{L^1\to L^1}||P^{n-i-1}_{\xi,T_2} f||_{L^1}.$$

Since $||P_{\xi,T_2} f||_{L^1} \leq ||f||_{L^1}$ and by Lemma 2.7.12, we have that

$$||P^n_{\xi,T_1} f - P^n_{\xi,T_2} f||_{L^1} \leq \sum_{i=0}^{n} C\theta^i ||T_1 - T_2||_\infty 5||\rho_\xi||_{BV}||f||_1$$

and the thesis follows. □

3 Proof of Theorem 1.1.6

The results in Sect. 2 already allow us to prove Theorem 1.1.6.

Proof of Theorem 1.1.6 Hypothesis R2 and R4 together with Lemma 2.5.11 prove that for all $\xi \geq \xi_1$ the operator P_ξ contracts the space of average 0 functions in BV (and equivalently in L^1). This guarantees uniqueness of the stationary measure.

Hypothesis R3 guarantees that $\ln(|T'|) \in L^p(m)$ for $p > 1$; by Corollary 2.6.5 together with Hypothesis R1 this allows us to prove that the function

$$\lambda(\xi) = \int_{-1}^{1} \ln(|T'|)d\mu_\xi$$

is well defined and continuous in $[0, +\infty)$.

Corollary 2.6.6 together with Hypothesis D2 and R3 allow us to state that

$$\lambda(0) > 0, \quad \lim_{\xi\to+\infty} \lambda(\xi) < 0,$$

therefore our system shows Noise Induced Order. □

4 Consequences for the Model

In this section, the noise kernel is

$$\rho(x) = \frac{1}{2}\chi_{[-1,1]}$$

the (normalized) characteristic function of the interval $[-1, 1]$.

The family $T_{\alpha,\beta} : [-1, 1] \to [-1, 1]$ is defined by

$$T_{\alpha,\beta}(x) = 1 - 2\beta|x|^{\alpha}. \tag{3}$$

4.1 Deterministic Behavior

The family $T_{\alpha,\beta}$ for $\alpha \geq 2$ and $0 < \beta \leq 1$ is a family of unimodal maps, a classical example of non-uniformly hyperbolic dynamics.

In this family, the prototypical example is the quadratic family, i.e., $T_{2,\beta}$ as β varies; the long term behavior of the system is strongly sensitive with respect to the parameter β: outside a parameter set of Lebesgue measure 0 (the infinitely renormalizable parameters [15]), the parameters can be classified into two categories:

- a dense subset of **regular parameters** where all the points converge to a periodic attracting orbit
- a positive measure Cantor set of **stochastic parameters** that admit an absolutely continuous invariant probability measure and have positive Lyapunov exponent.

It is worth discussing the properties of the Schwarzian derivative for the family $T_{\alpha,\beta}$.

Lemma 4.1.1 *For $\alpha > 1$ the Schwarzian derivative of $T_{\alpha,\beta}$ is well defined and negative in* $[-1, 0) \cup (0, 1]$.

Proof This follows from computation:

$$S(T_{\alpha,\beta}) = \left(\frac{T''_{\alpha,\beta}}{T'_{\alpha,\beta}}\right)' - \frac{1}{2}\left(\frac{T''_{\alpha,\beta}}{T'_{\alpha,\beta}}\right)^2.$$

For $x > 0$, we have that

$$\frac{T''_{\alpha,\beta}}{T'_{\alpha,\beta}}(x) = \frac{-2\beta\alpha(\alpha-1)x^{\alpha-2}}{-2\beta\alpha x^{\alpha-1}} = \frac{\alpha-1}{x}, \quad \frac{T'''_{\alpha,\beta}}{T'_{\alpha,\beta}}(x) = \frac{(\alpha-1)(\alpha-2)}{x^2}.$$

for $x < 0$, similarly we have that

$$\frac{T''_{\alpha,\beta}}{T'_{\alpha,\beta}}(x) = \frac{-2\beta\alpha(\alpha-1)(-x)^{\alpha-2}}{2\beta\alpha(-x)^{\alpha-1}} = -\frac{\alpha-1}{(-x)} = \frac{\alpha-1}{x}.$$

and

$$\frac{T'''_{\alpha,\beta}}{T'_{\alpha,\beta}}(x) = \frac{2\beta\alpha(\alpha-1)(\alpha-2)(-x)^{\alpha-3}}{2\beta\alpha(-x)^{\alpha-1}} = \frac{(\alpha-1)(\alpha-2)}{x^2}.$$

Therefore, for $x \in [-1, 0) \cup (0, 1]$ we have that

$$S(T_{\alpha,\beta})(x) = \frac{(\alpha-1)(\alpha-2)}{x^2} - \frac{3}{2}\frac{(\alpha-1)^2}{x^2} = -\frac{1}{2}\frac{\alpha^2-1}{x^2} < 0.$$

□

Remark 4.1.2 The unique point where the Schwarzian derivative of $T_{\alpha,\beta}$ is not defined is the critical point. This observation is not new, and is used extensively in [17]; intuitively, due to the slow recurrence of the critical orbit to the critical point, the levels of our tower are avoiding the critical point.

Therefore, when we build the induction scheme, the Schwarzian derivative of the iterates is going to be definite and negative.

We will prove now that our systems, when $\alpha \geq 2$ and $\beta = 1$ satisfy the hypothesis of [23, Theorem I.5]. This permits us to state that β is a density point of **stochastic** parameters, i.e., parameters that admit an a.c.i.p. and have positive Lyapunov exponent.

To avoid notation clutter, in some of the following equations we are going to use the notation $f_\beta(x) := f(\beta, x)$, and the notation $c_n(\beta) := f_\beta^n(0)$ for the critical orbit.

Definition 4.1.3 We say $f(\beta, x)$ is a **regular family** if

(1) $f(\beta, x)$ is C^2 in x, β;
(2) $c = 0$ is the unique critical point of $f(\beta, x)$, $f(\beta, x)$ is increasing on $[-1, 0)$, decreasing on $(0, 1]$, $c_2(\beta) < 0 < c_1(\beta)$ and $c_2(\beta) \leq c_3(\beta)$, and for all $x \in (-1, 0]$ we have that $f(\beta, x) > x$;
(3) there exists constants A_1^*, A_2^* and $\tau \geq 2$ such that for all β

$$A_1^* |x|^{\tau-1} \leq |D_x f_\beta(x)| \leq A_2^* |x|^{\tau-1}$$

and

$$\frac{|D_x f_\beta(x)|}{|D_x f_\beta(y)|} \leq \exp\left(C_* \left|\frac{x}{y} - 1\right|\right)$$

Lemma 4.1.4 *Fixed $\tilde{\alpha} \geq 2$ the family $f(\beta, x) := T_{\tilde{\alpha},\beta}(x)$ is a regular family.*

Proof Item (1), (2) and the first part of item (3) are trivial, the second part of item (3) follows from the fact that

$$(\alpha - 1) \ln\left(\frac{|x|}{|y|}\right) \leq (\alpha - 1)\left(\frac{|x|}{|y|} - 1\right).$$

□

Definition 4.1.5 A parameter β is called a **perturbable parameter** if there exists a constant $\varepsilon^* > 0$ such that

(1) for every $\delta \in (0, \epsilon^*)$ and $n \geq 1$, if $x \in I$ satisfies $f_\beta^i(x) \notin (-\delta, \delta)$ and $f_\beta^n(x) \in (-\delta, \delta)$ then $|(f_\beta^n)'(x)| \geq \epsilon^*$,
(2) for all $n \geq 1$, $c_n(\beta) \geq \epsilon^*$ and f_β has no stable periodic point,
(3) $\lim_{n\to+\infty} \partial_\beta f_\beta^n(c_0(\beta))/\partial_x f_\beta^{n-1}(c_1(\beta)) = Q^* \neq 0$.

Lemma 4.1.6 *Fixed $\tilde{\alpha} \geq 2$, if we denote by $f_\beta(x) := T_{\tilde{\alpha},\beta}(x)$, the parameter $\beta = 1$ is a perturbable parameter.*

Proof Item (2) in the definition of perturbable parameter is trivial, since $c_2(1) = -1$, which is a fixed point.

Item (3) follows from the chain rule for the derivative with respect to the parameter, i.e.,

$$\frac{\partial}{\partial\beta}(f(\beta, g(\beta, x))) = \frac{\partial f}{\partial\beta}(\beta, g(\beta, x)) + \frac{\partial f}{\partial x}(\beta, g(\beta, x))\frac{\partial g}{\partial\beta}(\beta, x).$$

This allows us to check item (3); by a straightforward computation we have that

$$\frac{\partial f}{\partial \beta}(1,-1)=2 \quad , \frac{\partial f}{\partial x}(1,-1)=2\alpha$$

and that

$$(\partial_\beta f_1^n)(0)=\partial_\beta f_1(c_{n-1}(1))+\partial_x f_1(c_{n-1}(1))\partial_\beta(f_1^{n-1}(0)),$$

since $c_2(1)=-1$, which is a fixed point, we have that

$$(\partial_\beta f_1^n)(0)=2+2\alpha\partial_\beta(f_1^{n-1}(0)),$$

which in turn tells us that

$$(\partial_\beta f_1^n)(0)\sim(2\alpha)^{n-1}$$

which in turn implies item (3).

The last condition we need to check is condition (1); we will follow a classical construction from [14].

We will denote by η the positive fixed point of $f_1(x)$; denote by f_L the left branch of $f_1(x)$ and by f_R the right branch. We will identify by a string of "R" and "L" the preimages of η through f_R and f_L, i.e.,

$$RLLL=f_R^{-1}(f_L^{-1}(f_L^{-1}(f_L^{-1}(\eta)));$$

we observe that $L=-\eta$. We will denote by L^k a sequence of k consecutive "L" and similarly for "R".

Outside of the domain $I=(-\eta,\eta)$ the map $f_1(x)$ is uniformly expanding. The preimages RL^k for $k=1,2,\dots$ are all bigger than η, so their left and right preimages fall in $(-\eta,\eta)$ when taking their left and right preimages LRL, RRL, $LRLL$, $RRLL$,... we obtain a countable partition of $(-\eta,\eta)$.

Denote by $\Delta_k=(RRL^{k+1},RRL^k)$ and by $\Delta_{-k}=(LRL^{k+1},LRL^k)$; observe that by construction Δ_k and Δ_{-k} are mapped diffeomorphically onto $(-\eta,\eta)$ by f_1^{k+1}. Moreover, on $[-1,0)\cup(0,1]$ we have that f_1 has negative Schwarzian derivative; this allows us to show that if x belongs to Δ_k, it will come back to $(-\eta,\eta)$ with derivative bigger than 1, by Koebe distortion lemma.

Now, if $x\notin I$, $f_1^k(x)\notin I$ for $k=1,\dots n$, $f_1^n(x)\in I$, since f_1 is uniformly expanding outside I, the condition is satisfied.

If $x\in I$, then x belongs to some Δ_i and if we denote by $r(x)=|i|+1$ the return time to I then x returns to I after $r(x)$ iterations and $|Df_1^{r(x)}(x)|>1$.

If coming back it enters $(-\delta,\delta)$, then the condition is satisfied; if it returns to $I\setminus(-\delta,\delta)$, then it will return to I only after $r(f_1^{r(x)}(x))$ steps, with derivative

$$|Df_1^{r(f_1^{r(x)}(x))}(x)|=|Df_1^{r(f_1^{r(x)}(x))}(f_1^r(x))Df_1^{r(x)}(x)|>1.$$

The only remaining case is when x starts outside I and then hits $I\setminus(-\delta,\delta)$ in k steps. In this case the modulus of the derivative is bigger than 1 before k and then we will need at least $r(f_1^k(x))$ steps to get back to I, guaranteeing that the derivative is bigger than 1. □

This allows us to use [23] to prove the following.

Theorem 4.1.7 (Theorem 1.5 [23]) *Let $\tilde{\alpha}\geq 2$ and let $f_\beta(x)=T_{\tilde{\alpha},\beta}(x)$; let $\beta=1$; there exists positive constants $C,\gamma,\lambda,\epsilon$ such that 1 is a density point for the set of parameters Ω such that*

(1) *f_β has no stable periodic point,*
(2) *for all $n \geq 1$, $|f_\beta^n(0)| > \epsilon \exp(-n\gamma)$,*
(3) *for all $n \geq 0$, $|(f_\beta^n)'(f_\beta(0))| > C \exp(n\gamma)$,*
(4) *for all $n \geq 1$, if $x \in [-1, 1]$ satisfies $f_\beta^k(x) \neq 0$ for all $k = 1, \ldots n-1$ and $f_\beta^n(x) = 0$, then $|(f_\beta^n)'(x)| \geq C \exp(n\lambda)$.*

This implies that $\beta = 1$ is a density point for the set of parameters that admit an absolutely continuous invariant measure and with positive Lyapunov exponent with respect to this measure.

Remark 4.1.8 As pointed out by one of the referees, the family $T_{\alpha,\beta}$ is not C^3 for $2 < \alpha < 3$, so many results as in [13, 22] do not apply in this interval of exponents. The results of [23] works under lower regularity conditions.

Indeed, many of the technical details in the next sections are needed to apply our theory to the maps $T_{\alpha,\beta}$ for $\alpha \in (2, 3)$. The treatment is simplified for systems with higher regularity.

4.2 Stochastic Stability

We remember that the noise is distributed uniformly, i.e., the mother noise kernel is

$$\rho(x) = \frac{1}{2}\chi_{[-1,1]}.$$

We are interested in answering the following question: if μ_0 is the invariant measure for the deterministic system and μ_ξ is the stationary measure for the random dynamical system with noise amplitude ξ, is it true that μ_ξ goes to μ_0 as the noise amplitude goes to 0? And in which sense does this happen, i.e., is it convergence in the weak-* topology, or we can have stronger statements on the convergence? This problem is called stochastic stability, and many results have appeared during the years [1, 2, 4, 5, 17, 22], where stochastic stability is proved under different hypothesis and regularity assumptions.

In [5] strong stochastic stability is proved that for C^4 unimodal maps with nondegenerate critical points, negative Schwarzian derivative and such that, if c is the critical point, there exists $\gamma > 0$, $\lambda_c > 1$, $H_0 \geq 1$, $e^{2\gamma} < \sqrt{\lambda_c}$ such that

- $|T^k(c)| \geq e^{-\gamma k}$ for all $k > H_0$
- $|(T^k)'(T(c))| \geq \lambda_c^k$ for all $k > H_0$
- f is topologically mixing on the interval bounded by c_1 and c_2.

This means that if f_ξ is the density of μ_ξ and f_0 is the invariant density of the a.c.i.p. of T, we have that f_ξ converges to f_0 in L^1 norm.

The argument goes as follows, the condition above allows the authors in [5] to construct a uniformly expanding[2] tower extension of the dynamic $\hat{T} : \hat{I} \to \hat{I}$, where $\hat{I} \subset \mathbb{N} \times [-1, 1]$ is the union of sets of the form $\{k\} \times B_k$ and the B_k's are a partition of full measure of $[-1, 1]$. If $\Pi(k, x) = x$ is the projection taking a point in $\{k\} \times B_k$, we have that $\Pi \circ \hat{T} = T \circ \Pi$.

This tower construction, as constructed in [5] works also for all deterministic perturbations $T(x) + \omega$ where $\omega < \epsilon_0$, so they are able to construct an extension of the random dynamical system $\hat{T}_\xi : \hat{I} \to \hat{I}$ such that $\Pi \circ \hat{T}_\xi = T_\xi \circ \Pi$, and using a perturbation argument prove the following theorem.

[2] with respect to an adapted Riemann metric by conjugating the Perron–Frobenius operator by multiplication with a cocycle.

Theorem 4.2.1 *There exists an $\xi_0 > 0$ such that for all $\xi \in [0, \xi_0)$ the random dynamical system $\hat{T}_\xi$ on $\hat{I}$ admits a unique stationary measure with density $\hat{f}_\xi$ in BV with respect to the Lebesgue measure $\hat{m}$ in $\hat{I}$. Moreover*

$$\lim_{\xi \to 0^+} ||\hat{f}_\xi - \hat{f}_0||_{BV} \to 0,$$

which implies that $f_\xi := \Pi_* \hat{f}_\xi$ converges to $f_0 := \Pi_* \hat{f}_0$ in L^1, $\mu_\xi = f_\xi dm$ is a stationary measure for T_ξ, $\mu_0 = f_0 dm$ is an invariant measure for T and, there exists $\tilde{C} > 0, 0 < \tilde{\theta} < 1$ such that for all $\xi \in [0, \xi_0)$

$$||P_\xi^n|_{\mathcal{U}_0}||_{BV} < \tilde{C}\tilde{\theta}^n.$$

We will not give a full proof of the Theorem, since it is quite a technical argument and the estimates can be done verbatim, but we will show where we can relax the hypothesis of negative Schwarzian derivative on the whole domain and the hypothesis that the map is C^4 on the whole domain.

Sketch of proof In the following, let $f(x) = T_{\alpha,\beta}(x)$, where β is a stochastic parameter obtained from Theorem 4.1.7. Without loss of generality, to avoid cluttering with constants, we assume the parameter satisfies

- $|c_n(\beta)| > e^{-\gamma n}$, for some small α, $n \geq 1$ (slow recurrence to the critical point),
- $|(f^n)'(c_1(\beta))| > \lambda_c^n$, for some $\lambda_c > 1$, for all $n \geq 1$ (expansivity along the critical orbit),

Following [17] p. 287 we fix $\lambda > 1$ and $\rho < e^\gamma$ such that

$$e^\gamma \lambda \rho \leq \lambda_c^{1/\alpha},$$

where α is the exponent of $T_{\alpha,\beta}$, and letting $\gamma < \beta_1 < \beta_2 < 2\gamma$, the condition above implies that

$$e^{\beta_i/2} \lambda \rho \leq \lambda_c^{1/\alpha}$$

for $i = 1, 2$.

Let $c_k = f^k(0)$, and for all $k > 0$ let $B_k = [a_k, b_k]$ be a set such that $[c_k - e^{-\beta_2 k}, c_k + e^{-\beta_2 k}] \supseteq B_k \supseteq [c_k - e^{-\beta_1 k}, c_k + e^{-\beta_1 k}]$; due to the slow recurrence to the critical point, we have that $0 \notin B_k$ for all $k > 0$; let $B_0 = [-1, 1]$.

We fix a small $\delta > 0$, to guarantee that once in the neighborhood $(-\delta, \delta)$ we will go up enough levels of the tower and, denoting by $f_t(x) = f(x) + t$ and, letting $E_k = B_k \times \{k\}$ for $k \geq 0$ and $\hat{I} = \bigcup_{k \geq 0} E_k$; we define $\hat{f}_t : \hat{I} \to \hat{I}$

$$\hat{f}_t(x, k) = \begin{cases} (f_t(x), k+1) & \text{if} k \geq 1 \text{and} f_t(x) \in B_{k+1} \\ (f_t(x), 1) & \text{if} k = 0 \text{and} x \in (-\delta, \delta) \\ (f_t(x), 0) & \text{otherwise.} \end{cases}$$

for all $t \in (-\epsilon_0, \epsilon_0)$ (with ϵ_0 small). The Π map is defined as $\Pi(x, k) = x$.

We define the unperturbed cocycle $\omega_0 : \hat{I} \to \mathbb{R}$

$$\omega_0(x, k) = \begin{cases} \frac{\lambda^k}{|(f^k)'(f_+^{-k}(x,k))|} & \text{if } (x, k) \in Im(\hat{f}^k) \\ 0 & \text{otherwise} \end{cases}$$

where $f_+^{-k}(x, k) = y$ is the unique point in $(0, \delta)$ such that $\hat{f}^k((y, 0)) = (x, k)$, we will not define the perturbed cocycle ω_ϵ since the negative Schwarzian derivative hypothesis enters into play only in the proof of the properties of ω_0

The negative Schwarzian derivative hypothesis is used only in [5] Lemma 4 and the Sublemma in Sect. 4.

We start by showing how to adapt the proof of [5, Lemma 4]. Note that the support of the cocycle ω_0 in E_k is an interval for each $k \geq 1$, with endpoints in the set $\partial E_k \cup \{\hat{f}^k(0,0), \hat{f}^k(\delta,0), \hat{f}^k(-\delta,0)\}$.

For $k \geq 1$ let the subintervals of E_k defined as $\beta_k^+ = \{(y,k) \mid f(y) > b_{k+1} - \epsilon\}$ and $\beta_k^- = \{(y,k) \mid f(y) < a_{k+1} + \epsilon\}$, and by γ_k^+ and γ_k^- respectively their intersection with the set $\{\omega_\epsilon(x,k) \neq 0\}$.

For $(y,k) \in \gamma_k^+$, and similarly for γ_k^- we have that

$$\frac{\omega_0(y,k)}{|f'(y)|} = \frac{\lambda^k}{|(f^{k+1})'(\hat{f}_+^{-k}(y,k))|},$$

remark that neither $\hat{f}_+^{-k}(\gamma_k^+)$ nor $\Pi(E_j \cap \mathrm{supp}(\omega_0))$ contain 0 (refer to the proof [5, Lemma 4], Line 5), so the Schwarzian derivative of $f|_{\hat{f}_+^{-k}(\gamma_k^+)}$ is defined and negative and similarly for all its iterates. Therefore $|(f^{k+1})'(\hat{f}_+^{-k}(y,k))|$ has a unique maximum and Lemma 4 follows under our weaker hypothesis, by exchanging the order of the arguments in Line 4 and Line 5.

A similar argument works for the Sublemma in [5, Sect. 4], above equation 4.3, since f_t^n has no critical points in $\underline{\gamma}$, and the point where the Schwarzian derivative is not defined correspond to the critical points, the Schwarzian derivative of f_t^n is defined and negative, implying that the function $g^{(n)}$ has at most a local minimum on $\underline{\gamma}$.

We need to assess the lack of full C^4 regularity; the only place where the C^4 regularity of the map is used is in [5, "Climbing the tower" p. 497], to prove the regularity of the function $K(x)$ defined as

$$K(x) = \frac{|f'(x_-)|}{|f'(x)|},$$

where x_- is the unique point with $x_- \neq x$ and $f(x) = f(x_-)$. We need to prove that there exists finite constants K and $\tilde{K}$ such that

$$\sup_{x \neq 0} K(x) \leq K, \quad \mathrm{Var}_{x \neq 0}(K(x)) \leq \tilde{K}.$$

Remark that in our family, we have that $K(x) \equiv 1$ for all $x \neq 0$; so these are trivially satisfied.

The proof then follows directly from the estimates in [5]. □

Remark 4.2.2 While we fixed the uniform noise kernel, the class of noise kernels for which the result in [5] holds is larger: in our framework of rescaled noise $\rho_\xi(x) = \rho(x/\xi)/\xi$ they can be restated as the fact that ρ is bounded (which follows from Bounded Variation) and the fact that, if we denote by $J = \{t \mid \rho(t) > 0\}$, $0 \in J$ and $\ln(\rho|_J)$ is concave.

This has an important consequence, i.e., continuity of the Lyapunov exponent near 0.

Corollary 4.2.3 *In the hypothesis of Theorem 4.2.1, letting* $T(x) = T_{\alpha,\beta}(x)$

$$\lim_{\xi \to 0^+} \int_{-1}^{1} \ln(|T'|) f_\xi dm = \int_{-1}^{1} \ln(|T'|) f_0 dm.$$

Proof By direct computation

$$\left|\int_{-1}^{1} \ln(|T'|)(f_\xi - f_0)dm\right| = \left|\int_{-1}^{1} \ln(|T'|)\Pi_*(f_\xi - f_0)dm\right|$$
$$= \left|\sum_{k\in\mathbb{N}} \int_{\{k\}\times B_k} \ln(|T'(\Pi(\hat{x})|)(\hat{f}_\xi - \hat{f}_0)d\hat{m}(x)\right| \leq ||\ln(|T'|)||_{L^1([-1,1])}||\hat{f}_\xi - \hat{f}_0||_{BV(\hat{I})},$$

where $\hat{m}$ is the Lebesgue measure on $\hat{I}$, which implies the thesis since $\hat{f}_\xi$ converges to $\hat{f}_0$ in $BV(\hat{I})$. □

Corollary 4.2.4 (Corollary of [5]) *Let $T_{\alpha,\beta}$; fix $\alpha \geq 2$ and let the mother kernel be $\rho(x) = \chi_{[-1,1]}$, i.e, the noise in our random dynamical system is the uniform noise. Then $\beta = 1$ is a density point for the set of parameters Ω for which there exists a $\xi_0 > 0$ such that:*

(1) *for all $\xi \in [0, \xi_0)$ there exists a unique stationary measure μ_ξ,*
(2) *the density of the stationary measure μ_ξ converges to the density of the deterministic system in L^1 as ξ goes to 0 (strong stochastic stability),*
(3) *$\int_{-1}^{1} \ln(|T'_{\alpha,\beta}|)d\mu_\xi$ is a continuous function of the noise amplitude in $[0, \xi_0)$,*
(4) *there exists $\tilde{C} > 0$, $0 < \tilde{\theta} < 1$ such that for all $\xi \in [0, \xi_0)$*

$$||P_\xi^n||_{BV} \leq \tilde{C}\tilde{\theta}^n.$$

In particular, hypothesis D1, D2, R1, R2 and R4 of Theorem 1.1.6 are satisfied.

Remark 4.2.5 All the arguments presented in the sketch of the proof above are already known in literature, see [17].

We prove now that hypothesis D3 is also satisfied.

Lemma 4.2.6 *For $\alpha \in [2, +\infty)$, $\beta \in (0, 1]$*

$$\ln(|T'_{\alpha,\beta}|) \in L^p([-1, 1]),$$

for all $p \geq 1$.

Proof Follows by computation; let $x < 0$, the $x > 0$ case is analogous.

$$T'_{\alpha,\beta}(x) = 2\beta\alpha(-x)^{\alpha-1},$$

therefore

$$\ln(|T'_{\alpha,\beta}|) = \ln(2) + \ln(\beta) + \ln(\alpha) + (\alpha - 1)\ln(|x|),$$

which is in $L^p([-1, 1])$ for all $p \geq 1$ since $\ln(|x|)$ is in $L^p([-1, 1])$ for all $p \geq 1$. □

We need now to identify under which conditions hypothesis R3 is satisfied.

4.3 Large Noise Limit

By corollary 2.5.3 as the amplitude of the noise ξ grows, we have that f_ξ converges to the uniform density on $[-1, 1]$.

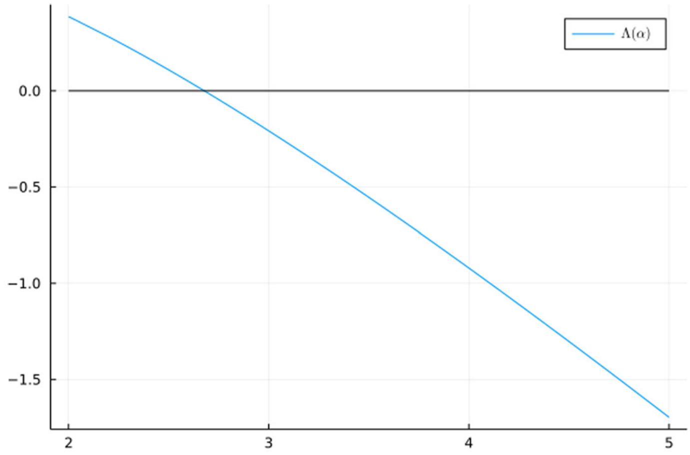


Fig. 2 The graph of $\Lambda(\alpha)$

Fixed $\beta = 1$ we define the following function, the large noise limit of the Lyapunov exponent of $T_{\alpha,\beta}$:

$$\Lambda(\alpha) = \int_{-1}^{1} \ln(|T'_\alpha|)\frac{dm}{2} = \ln(2) + \ln(\alpha) + 1 - \alpha$$

This is a decreasing function of α, for $\alpha \geq 2$, moreover $\Lambda(2) > 0$ and the function Λ has a zero $\tilde{\alpha}$ contained in the interval $[2.67834, 2.67835]$ [3]. A plot of Λ is found in Fig. 2.

Corollary 4.3.1 *For $\alpha > \tilde{\alpha}$, the map $T_{\alpha,1}$ presents Noise Induced Order.*

Proof By Corollary 4.2.4, hypothesis D1, D2, R1, R2, R4 are satisfied for $T_{\alpha,1}$. By Lemma 4.2.6 hypothesis D3 is satisfied. If $\alpha > \tilde{\alpha}$ hypothesis R3 is also satisfied, and by Theorem 1.1.6 we have the thesis. □

4.4 Behavior as the Parameter β Varies

In this section we study the behavior of the Lyapunov exponent of $T_{\alpha,\beta}$ in presence of noise, when we fix α and vary β.

We extend the large noise amplitude limit function to allow also β to vary; by a simple computation

$$\Lambda(\alpha, \beta) := \int_{-1}^{1} \ln(|T'_{\alpha,\beta}|)dm = \ln(2) + \ln(\alpha) + 1 - \alpha + \ln(\beta).$$

Since β belongs to $(0, 1]$, this is an increasing function of β, so, if $\alpha > \tilde{\alpha}$ and $\beta < 1$ we have that the Lyapunov exponent for big noise sizes is negative.

Lemma 4.4.1 *Let $T_{\alpha,\beta}$ be the map defined in Eq. (3). Then, for $h > 0$ we have that*

$$|T_{\alpha,\beta+h}(x) - T_{\alpha,\beta}(x)| \leq 2h|x|^\alpha \leq 2h.$$

[3] obtained with Julia ValidatedNumerics package.

Proof We will prove the inequality on [0, 1], the conclusion follows by symmetry:

$$|2(\beta+h)x^{\alpha}-2\beta x^{\alpha}| \leq 2h|x|^{\alpha}.$$

□

Remark 4.4.2 It is possible to compute an estimate also as α varies, proving the inequality on [0, 1], the conclusion follows by symmetry:

$$|2(\beta+h)x^{\alpha+k}-2\beta x^{\alpha}| \leq |2(\beta+h)x^{\alpha+k}-2\beta x^{\alpha+k}|+|2\beta x^{\alpha+k}-2\beta x^{\alpha}|;$$

we focus now on

$$|2\beta x^{\alpha+k}-2\beta x^{\alpha}| = k2\beta|x|^{\alpha}\frac{|x|^{k}-1}{k};$$

the inequality is written in this specific form

$$k|x|^{\alpha}\frac{|x|^{k}-1}{k} = k|x|^{\alpha}\ln|x|+O(k^{2}),$$

therefore, for small k we have that, for some constant C

$$||T_{\alpha+k,\beta+h}-T_{\alpha,\beta}||_{\infty} \leq |h|+C|k|,$$

since $|x|^{\alpha}\ln(|x|)$ goes to 0 for $x \to 0$ and has bounded derivative in [0, 1] for any $\alpha > 1$.

Definition 4.4.3 Fix $\alpha \geq 2$. In the following we will denote by $L_{\xi,\beta}$ the annealed transfer operator of $T_{\alpha,\beta}$ with noise amplitude ξ, $P_{\xi,\beta}$ the associated annealed Perron–Frobenius operator. If a unique stationary measure exists, we will denote it by $\mu_{\xi,\beta}$ and its density by $f_{\xi,\beta}$.

Corollary 4.4.4 *Suppose there exist β,ξ such that $P_{\xi,\beta}$ contracts the space of average 0 function in L^1; then there exists an $\epsilon > 0$ such that for all $0 < h < \epsilon$ the operator $P_{\xi,\beta+h}$ contracts the space of average 0 functions in L^1.*

Proof This follows from Lemmas 2.7.13 and 4.4.1; if C,θ are the contraction constants of $P_{\xi,\beta}$ then

$$\begin{aligned}||P^{n}_{\xi,\beta+h}|\mathcal{U}_0||_{L^1\to L^1} &\leq ||P^{n}_{\xi,\beta}|\mathcal{U}_0||_{L^1\to L^1}+||P^{n}_{\xi,\beta+h}-P^{n}_{\xi,\beta}||_{L^1\to L^1}\\ &\leq C\theta^{n}+3\frac{C}{1-\theta}h||\rho_{\xi}||_{BV}.\end{aligned}$$

If N is such that $C\theta^{N} < 1$ and h is small enough, this implies that

$$||P^{n}_{\xi,\beta+h}|\mathcal{U}_0||_{L^1\to L^1} < 1.$$

□

Corollary 4.4.5 *Suppose there exist β,ξ such that $P_{\xi,\beta}$ contracts the space of average 0 function in L^1 with constants C,θ; then if h is small enough*

$$||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^1} \leq 6h||\rho_\xi||_{BV}\frac{C}{1-\theta}$$

Proof From Corollary 4.4.4, we get that $P_{\xi,\beta+h}$ contracts the space of average 0 functions in L^1, therefore there exists a unique stationary density for $P_{\xi,\beta+h}$.

Let N such that $C\theta^N < 1/2$, then

$$\begin{aligned}||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^1} &\leq ||P^n_{\xi,\beta}(f_{\xi,\beta+h} - f_{\xi,\beta})||_{L^1} + ||(P^n_{\xi,\beta} - P^n_{\xi,\beta+k})f_{\xi,\beta+h}||_{L^1}\\ &\leq \frac{1}{2}||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^1} + ||(P^n_{\xi,\beta} - P^n_{\xi,\beta+k})f_{\xi,\beta+h}||_{L^1}\end{aligned}$$

as in the proof of Lemma 2.6.1, and therefore, since $||f_{\xi,\beta+h}||_{L^1} = 1$ we have

$$||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^1} \leq 6h||\rho_\xi||_{BV}\frac{C}{1-\theta}.$$

□

Lemma 4.4.6 *Fix* $\alpha \geq 2$ *and let*

$$\lambda_\xi(\beta) := \int_{-1}^{1} \ln(|T'_{\alpha,\beta}|)f_{\xi,\beta}dm.$$

If P_{ξ_0,β_0} *contracts the space of average* 0 *functions in* L^1*, then the function* λ_{ξ_0} *is defined at* β_0 *and it is Hölder continuous with respect to* β *at* β_0.

Proof We observe now that, by Hölder inequality

$$||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^r} \leq (||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^1})^{1/r}(||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^\infty})^{1-1/r},$$

and that

$$||f_{\xi,\beta+h} - f_{\xi,\beta}||_{L^\infty} \leq 6||\rho_\xi||_{BV}$$

Since $\ln|x|$ is in $L^p([-1, 1])$ for $p > 1$, the result follows. □

Remark 4.4.7 Under stronger hypothesis on the noise kernel it is possible to prove further regularity results on

$$\lambda(\xi, \alpha, \beta) = \int_{-1}^{1} \ln(|T'_{\alpha,\beta}|)d\mu_{\xi,\alpha,\beta},$$

where the function above is defined if there is a unique stationary measure for $\mu_{\xi,\alpha,\beta}$ for the annealed transfer operator of $T_{\alpha,\beta}$, using the linear response theory for random dynamical systems developed in [10, 12].

Corollary 4.4.8 *Fixed* $\alpha \geq \tilde{\alpha}$ *there exists an* $\epsilon(\alpha)$ *such that for all* $\beta \in (1 - \epsilon(\alpha), 1]$ *the map* $T_{\alpha,\beta}$ *presents Noise Induced Order.*

Proof Let $\beta = 1$. This is the full branch case, by the results in Sect. 4.2 we know that there exists an interval $[0, \xi_0)$ and $C > 0$, $0 < \theta < 1$ such that for all $\xi \in [0, \xi_0)$, we have that $||P^n_{\xi,1}|_{\mathcal{U}_0}|| \leq C\theta^n$.

Therefore the stationary measure $\mu_{\xi,1}$ is unique for all $\xi \in [0, \xi_0)$ and by Sect. 4.2 there is a ξ_1 such that for all $\xi \in [0, \xi_1]$ we have that

$$\int_{-1}^{1} \ln(|T'_{\alpha,1})d\mu_{\xi,1} > 0.$$

Let $\hat{\xi} = \min(\xi_0, \xi_1)$; fix a $\xi \in [0, \hat{\xi})$, and let ϵ_0 such that for all $h < \epsilon_0$ the operator $P_{\xi,1-h}$ contracts the space of average 0 functions in L^1; this ϵ_0 exists by Corollary 4.4.4 and depends on α. Then by Corollary 4.4.5, we have that for all $h \in [0, \epsilon_0)$

$$\begin{aligned}
&\left|\int_{-1}^{1} \ln(|T'_{\alpha,1}|) f_{\xi,1} dm - \int_{-1}^{1} \ln(|T'_{\alpha,1-h}|) f_{\xi,1-h} dm\right| \\
&\leq \left|\int_{-1}^{1} \ln(|T'_{\alpha,1}|) f_{\xi,1} dm - \int_{-1}^{1} \ln(|T'_{\alpha,1-h}|) f_{\xi,1} dm\right| \\
&\quad + \left|\int_{-1}^{1} \ln(|T'_{\alpha,1-h}|) f_{\xi,1} dm - \int_{-1}^{1} \ln(|T'_{\alpha,1-h}|) f_{\xi,1-h} dm\right| \\
&\leq \ln(1-h) + 2h||\ln(|T'_{\alpha,1-h}|)||_{L^1} 3||\rho_\xi||_{BV} \frac{C}{1-\theta}.
\end{aligned}$$

Therefore, for the ξ fixed above there exists an $\epsilon_1 < \epsilon_0$, depending on α, such that for all $h \in [0, \epsilon_1)$

$$\int_{-1}^{1} \ln(|T'_{\alpha,1-h}|) f_{\xi,1-h} dm > 0.$$

Recall now that the big noise amplitude limit of the Lyapunov exponent for $T_{\alpha,1-h}$ is given by

$$\int_{-1}^{1} \ln(|T'_{\alpha,1-h}|) \frac{dm}{2} = \ln(2) + \ln(\alpha) + 1 - \alpha + \ln(1-h);$$

therefore, if $\alpha > \tilde{\alpha}$, there exists an ϵ_2 (depending on α) such that for all $h \in [0, \epsilon_2)$ the big noise amplitude limit is negative.

Let $\epsilon = \min(\epsilon_1, \epsilon_2)$ then, for all $\beta \in (1-\epsilon, 1]$ the Lyapunov exponent at noise amplitude ξ is positive and the big noise amplitude limit is negative, therefore, we have Noise Induced Order. □

Acknowledgements The author thanks Y. Sato, M. Benedicks, M. Monge and S. Galatolo for introducing the problem, pointing in the right direction and giving many of the tools. The author thanks warmly E. Ghys and A. Blumenthal for reading the paper, posing questions and providing ideas. The author thanks the anonymous referees, whose questions and comments led to a rewrite of the article, which we hope is clearer. This paper is dedicated to W. Tucker in occasion of his 50th birthday. The author thanks the ICTP for the hospitality and was partially supported by CNPq, University of Uppsala and KAW Grant 2013.0315. UFRJ, CAPES (through the programs PROEX and the CAPES-STINT project "Contemporary topics in non uniformly hyperbolic dynamics"). The author is currently under "Afastamento do país para qualificação profissional, apresentação de trabalhos técnico-cientìficos e colaboração institucional do pessoal docente e técnico-administrativo" from UFRJ and is currently a Specially Appointed Associate Professor at Hokkaido University. The author would like to thank Prof. Hiroki Sumi and Kyoto university for their hospitality during the final revision of this article.

Data availability The script used to produce the data, the data, and a Jupyter notebook used for the plot in Fig. 1 can be found at https://github.com/orkolorko/UnimodalNIO.

Declarations

Conflict of interest The author has no competing interests to declare that are relevant to the content of this article.

References

1. Alves, J.F., Araújo, V.: Random perturbations of nonuniformly expanding maps. In: de Melo, W., Viana, M., Yoccoz, J.-C. (eds.) Geometric methods in dynamics (I): Volume in honor of Jacob Palis, number 286 in Astérisque, Société mathématique de France (2003)
2. Alves, J.F., Vilarinho, H.: Strong stochastic stability for non-uniformly expanding maps. Ergod. Theory Dyn. Syst. **33**(3), 647–692 (2013)
3. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs (2000)
4. Araujo, V., Pacifico, M.J., Pinheiro, M.: Adapted random perturbations for non-uniformly expanding maps. Stoch. Dyn. **14**(04), 1450007 (2014)
5. Baladi, V., Viana, M.: Strong stochastic stability and rate of mixing for unimodal maps. Ann. Sci. de l'Ecole Norm. Superieure. Ser. **29**(4), 483–517 (1996)
6. Blumenthal, A., Xue, J., Young, L.-S.: Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. Ann. Math. **185**(1), 285–310 (2017)
7. Blumenthal, A., Xue, J., Young, L.-S.: Lyapunov exponents and correlation decay for random perturbations of some prototypical 2d maps. Commun. Math. Phys. **359**(1), 347–373 (2018)
8. Cherubini, A.M., Lamb, J.S.W., Rasmussen, M., Sato, Y.: A random dynamical systems perspective on stochastic resonance. Nonlinearity **30**(7), 2835–2853 (2017)
9. Chihara, T., Sato, Y., Nisoli, I., Galatolo, S.: Existence of multiple noise-induced transitions in Lasota-Mackey maps. Chaos **32**(1), 013117 (2022)
10. Galatolo, S., Giulietti, P.: A linear response for dynamical systems with additive noise. Nonlinearity **32**(6), 2269–2301 (2019)
11. Galatolo, S., Monge, M., Nisoli, I.: Existence of noise induced order, a computer aided proof. Nonlinearity **33**(9), 4237–4276 (2020)
12. Galatolo, S., Sedro, J.: Quadratic response of random and deterministic dynamical systems. Chaos **30**(2), 023113 (2020)
13. Gao, B., Shen, W.: Summability implies Collet-Eckmann almost surely. Ergod. Theory Dyn. Syst. **34**(4), 1184–1209 (2014)
14. Jakobson, M.V.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. **81**, 39–88 (1981)
15. Lyubich, M.: Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture. Ann. Math. **149**(2), 319–420 (1999)
16. Matsumoto, K., Tsuda, I.: Noise-induced order. J. Stat. Phys. **31**(1), 87–106 (1983)
17. Metzger, R.J.: Stochastic stability for contracting Lorenz maps and flows. Commun. Math. Phys. **212**(2), 277–296 (2000)
18. Meyn, S., Tweedie, R.L.: Markov Chains and Stochastic Stability, 2nd edn. Cambridge University Press, Cambridge (2009)
19. RyanRogers, U.: Does weak convergence with uniformly bounded densities imply absolute continuity of the limit? Mathematics Stack Exchange. https://math.stackexchange.com/q/574130 (version: 2013-12-17)
20. Sarig, O.: Introduction to the transfer operator method. https://www.weizmann.ac.il/math/sarigo/sites/math.sarigo/files/uploads/transferoperatorcourse-bonn.pdf (2020)
21. Sato, Y., Klages, R.: Anomalous diffusion in random dynamical systems. Phys. Rev. Lett. **122**, 174101 (2019)
22. Shen, W.: On stochastic stability of non-uniformly expanding interval maps. Proc. Lond. Math. Soc. **107**(5), 1091–1134 (2013)
23. Thieullen, P., Tresser, C., Young, L.: Positive lyapunov exponent for generic one-parameter families of unimodal maps. J. Anal. Math. **64**(1), 121–172 (1994)

24. Viana, M.: A stochastic view of dynamical systems. http://www.im.ufrj.br/~coloquiomea/apresentacoes/viana.pdf, COLMEA, UFRJ (2009)
25. Viana, M.: Lectures on Lyapunov Exponents. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2014)