AN ELEMENTARY WAY TO RIGOROUSLY ESTIMATE CONVERGENCE TO EQUILIBRIUM AND ESCAPE RATES

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Abstract. We show an elementary method to obtain (finite time and asymptotic) computer assisted explicit upper bounds on convergence to equilibrium (decay of correlations) and escape rates for systems satisfying a Lasota Yorke inequality. The bounds are deduced from the ones of suitable approximations of the system's transfer operator. We also present some rigorous experiments on some nontrivial example.

Contents

1. **Introduction.** The evolution of a chaotic system is unpredictable and difficult to describe, but its statistical properties are sometime predictable and (reasonably) simple to describe. Many of these properties are related to its associated transfer operator, essentially via its spectrum. This is a linear operator representing the action of the dynamics on measures space. Let us consider the set $SM(X)$ of finite signed Borel measures on the metric space X. A Borel map $T: X \to X$ naturally induces a linear operator $L_T : SM(X) \to SM(X)$ called the **transfer operator**, defined as follows. If $\mu \in SM(X)$ then $L_T[\mu] \in SM(X)$ is the measure such that

$$L_T[\mu](A) = \mu(T^{-1}(A)).$$

Sometimes, when no confusion arises, we simply denote L_T by L.

In this paper we address two important statistical features of the dynamics: the rate of *convergence to equilibrium* and the *escape rate* in open dynamical systems.

Convergence to equilibrium (see Section 2) is a quantitative estimation of the speed in which a starting, absolutely continuous measure approaches the physical invariant measure. This is related to the spectrum of the transfer operator, since the speed is exponential in the presence of a spectral gap, and to decay of correlations. Indeed, an upper bound on the decay of correlations can be obtained from convergence to equilibrium estimation in a large class of cases, see [1]). An estimation for these rates is a key step in deducing many other consequences: central limit theorem, hitting times, recurrence rate, etc. (see, e.g., [8, 6, 13, 21]).

The escape rate refers to open systems, where the phase space has a "hole" (see Section 3) and one wants to understand, quantitatively, the speed of loss of mass of the system through the hole. This is related to the spectral radius of a truncated transfer operator and to the presence of metastable states (see, e.g., the book [2] for an introduction).

We will present a method which allows us to obtain an efficient, effective and quite elementary *finite time* and *asymptotic* upper estimations for these decay rates. The strategy is applicable under two main assumptions on the transfer operator.

- the transfer operator is regularizing on a suitable space; it satisfies a Lasota Yorke inequality (see Equation 2);

- the transfer operator can be approximated in a satisfactory way by a finite dimensional one. (see Equation 3)

The first item in some sense describes the small scale behavior of the system. The regularizing action implies that at a small scale we see a kind of uniform behavior. The macroscopic behavior is then described by a "finite resolution" approximation of the transfer operator. This will be represented by suitable matrices, as the transfer operator is linear, and its main features will be computable from the coefficients of the matrix.

In the following we will enter in the details of how this strategy can be implemented in general, and in some particular system for which we will present some experiment rigorously implemented by interval arithmetics.

2. **Recursive convergence to equilibrium estimation for maps satisfying a Lasota Yorke inequality.** Consider two vector subspaces of the space of signed measures on X

$$\mathcal{S} \subseteq \mathcal{W} \subseteq SM(X),$$

endowed with two norms, the strong norm $|| \ ||_s$ on $\mathcal{S}$ and the weak norm $|| \ ||_w$ on $\mathcal{W}$, such that $|| \ ||_s \geq || \ ||_w$ on $\mathcal{W}$.

We say that the probability measure preserving transformation (X, T, μ) has convergence to equilibrium with speed Φ with respect to these norms if for any Borel probability measure ν on X

$$||L^n \nu - \mu||_w \leq ||\nu||_s \Phi(n). \tag{1}$$

This speed can be also estimated by the rate of convergence to 0 of $||L^n \xi||_w$ for signed measures $\xi \in \mathcal{S}$ such that $\xi(X) = 0$ or by estimating the decay rate of correlation integrals like

$$\left| \int f \circ T^n \ g \ d\mu - \int f \ d\mu \int g \ d\mu \right|$$

for observables f, g in suitable function spaces.

It is important both to have a certified quantitative estimation for this convergence at a given time (numerical purposes, rigorous computation of the invariant measure as in [14],[15],[17], see also Remark 4), or an estimation for its asymptotic speed of convergence (computer assisted proofs of the speed of decay of correlations and its statistical consequences).

In the literature, the problem of computing rigorous bounds on the decay of correlation and convergence to equilibrium rate of a given system was approached by spectral stability results (see, e.g., [20]). The use of these methods is limited by the complexity of the assumptions and of the a priori estimations which are needed.

In the following we show how the Lasota Yorke inequality, which can be established in many systems, coupled with a suitable approximation of the system by a finite dimensional one allows us to directly deduce finite time and asymptotic upper bounds on the convergence to equilibrium of the system in a simple and elementary way.

Assumptions. *Let us suppose that our system satisfies the following:*

- *The system satisfies a Lasota Yorke inequality.*
  There exists constants $A, B, \lambda_1 \in \mathbb{R}$ with $\lambda_1 < 1$ such that $\forall f \in \mathcal{S}, \forall n \geq 1$

$$||L^n f||_s \leq A\lambda_1^n ||f||_s + B||f||_w. \tag{2}$$

- *There exists a family of "simpler" transfer operators L_δ approximating L satisfying a certain approximation inequality: there are constants C, D such that $\forall g \in \mathcal{S}, \forall n \geq 0$:*

$$||(L_\delta^n - L^n)g||_w \leq \delta(C||g||_s + nD||g||_w). \tag{3}$$

- *There exists $\delta > 0$, $\lambda_2 < 1$ and n_1 such that, setting*

$$V = \{\mu \in \mathcal{S} | \mu(X) = 0\}$$

we have $L_\delta(V) \subseteq V$ and

$$\forall v \in V, \ ||L_\delta^{n_1}(v)||_w \leq \lambda_2 ||v||_w. \tag{4}$$

Remark 1. In the following we will consider examples of systems satisfying such inequality, where $\mathcal{S}$ is the space of measures having a bounded variation density, $|| \ ||_s$ is the bounded variation norm and $|| \ ||_w$ is the L^1 one.

We also remark that condition (3) is natural for approximating operators L_δ defined (as it is commonly used), by $\pi_\delta L \pi_\delta$, where π_δ is a projection on a finite dimensional space with suitable properties, see Section 6.3, where it is shown how to obtain (3) under these assumptions.

We remark that in Equation (4), the condition is supposed on L_δ^n, which is supposed to be simpler than L^n (e.g., a discretization with a grid of size δ represented by a matrix, see Section 5) and its properties might be checked by some feasible computation.

Under the above conditions we can effectively estimate from above the convergence to equilibrium in the system in terms of the matrix

$$M = \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix}.$$

Since M is positive, its largest eigenvalue is

$$\rho = \frac{A\lambda_1^{n_1} + \delta n_1 D + \lambda_2 + \sqrt{(A\lambda_1^{n_1} - \delta n_1 D - \lambda_2)^2 + 4\delta BC}}{2}. \tag{5}$$

Let (a, b) be the left eigenvector of the matrix M associated to the eigenvalue ρ, normalized in a way that $a + b = 1$.

The condition $\rho < 1$ below implies that the powers of M go to zero exponentially fast. Note that the quantities δBC, $\delta n_1 D$ have a chance to be small when δ is small, but this is not guaranteed since the choice of n_1 depends on δ. If we consider the case of piecewise expanding maps, with L_δ being the Ulam approximation of L, this is the case (see [12], Theorem 12).

Theorem 2. *Under the previous assumptions 2, 3, 4, if $\rho < 1$ then for any $g \in V$,*
(i) the iterates of $L^{in_1}(g)$ are bounded by

$$\begin{pmatrix} ||L^{in_1}(g)||_s \\ ||L^{in_1}(g)||_w \end{pmatrix} \preceq M^i \begin{pmatrix} ||g||_s \\ ||g||_w \end{pmatrix}$$

Here $\preceq$ indicates the componentwise $\leq$ relation (both coordinates are less or equal).
(ii) In particular we have

$$||L^{in_1}g||_s \leq (1/a)\rho^i ||g||_s,$$
$$||L^{in_1}g||_w \leq (1/b)\rho^i ||g||_s.$$

(iii) Finally $\forall \, k \in \mathbb{N}$

$$||L^k g||_s \leq (A/a + B/b)\rho^{\left\lfloor \frac{k}{n_1} \right\rfloor} ||g||_s, \tag{6}$$

$$||L^k g||_w \leq (B/b)\rho^{\left\lfloor \frac{k}{n_1} \right\rfloor} ||g||_s. \tag{7}$$

Proof. (i) Let us consider $g_0 \in V$ and denote $g_{i+1} = L^{n_1}g_i$. By assumption 2 we have

$$||L^{n_1}g_i||_s \leq A\lambda_1^{n_1}||g_i||_s + B||g_i||_w.$$

Putting together the above assumptions 3 and 4 we get

$$||L^{n_1}g_i||_w \leq ||L_\delta^{n_1}g_i||_w + \delta(C||g_i||_s + n_1 D||g_i||_w)$$
$$\leq \lambda_2||g_i||_w + \delta(C||g_i||_s + n_1 D||g_i||_w). \tag{8}$$

Compacting these two inequalities into a vector notation, setting $v_i = \begin{pmatrix} ||g_i||_s \\ ||g_i||_w \end{pmatrix}$ we get

$$v_{i+1} \preceq \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix} v_i. \tag{9}$$

The relation $\preceq$ can be used because the matrix is positive. This proves (i) by an immediate induction.

(ii) Let us introduce the (a, b) balanced-norm as $||g||_{(a,b)} = a||g||_s + b||g||_w$. The first assertion gives

$$||L^{in_1}g||_{(a,b)} \leq (a, b) \cdot M^i \cdot \begin{pmatrix} ||g||_s \\ ||g||_w \end{pmatrix},$$

hence

$$||L^{in_1}g||_{(a,b)} \leq \rho^i ||g||_{(a,b)}.$$

From this, the statement follows directly.

(iii) Writing any integer $k = in_1 + j$ with $0 \leq j < n_1$, by assumption 2 we have

$$||L^k(g_0)||_s \leq A\lambda_1^j ||L^{in_1}g_0||_s + B||L^{in_1}g_0||_w,$$

and the conclusion follows by the second assertion (ii). □

Remark 3. We remark that our approach being based on a vector inequality has some similarity with the technique proposed in [15]. The first inequality used is the same in both approaches; the second is different. Our inequality relies on the approximation procedure and is more general.

Remark 4. We also remark that our method allows us to bound the strong norm of the iterates of a zero average starting measure. Considering $L^n(m - \mu)$ where m is a suitable starting measure (Lebesgue measure in many cases) and μ is the invariant one, we can understand how many iterations of m are necessary to arrive at a given small (strong) distance from the invariant one. This, added to a way to simulate iterations of L with small errors in the strong norm, allows in principle the rigorous computation of the invariant measure up to small errors in the strong norm.

3. **Escape rates.** A system with a hole is a system where there is a subset H such that when a point falls in it, its dynamics stops there. We consider H to be not part of the set where the dynamics acts. Iterating a measure by the dynamics will lead to losing some of the measure in the hole at each iteration, letting the remaining measure decay at a certain rate.

In many of such systems a Lasota Yorke and an approximation inequality (Equations 2, 3) can be proved, hence the procedure of the previous section allows us to estimate this rate, which is related to the spectral radius of the transfer operator.

Let us hence consider a starting system without hole with transfer operator L, consider a hole in the set H, and let 1_{H^c} be the indicator function of the complement of H. The transfer operator of the system with hole is given by

$$L_H f = 1_{H^c} L f.$$

Similar to the convergence to equilibrium, we can say that the escape rate of the system with respect to norms $|| \ ||_s, || \ ||_w$ is faster than Ψ if for each f:

$$||L^n f||_w \leq \Psi(n)||f||_s.$$

We will see in the next proposition that an iterative procedure as the one of the previous section can be implemented to estimate the escape rate. We will need assumptions similar to the ones listed before. These are natural assumptions for open systems constructed from a system satisfying a Lasota Yorke inequality. We will verify it in some example of piecewise expanding maps with a hole (see Section 5.2).

Theorem 5. *With the notations of Theorem 2 let us suppose*

- *the system with hole satisfies for some $\lambda_1 < 1$*

$$\forall\, n \geq 1, \quad ||L_H^n f||_s \leq A\lambda_1^n ||f||_s + B||f||_w. \tag{10}$$

- *$L_\delta(\mathcal{S}) \subseteq \mathcal{S}$ and there is an approximation inequality: $\forall\, g \in \mathcal{S}$*

$$\forall\, n \geq 1, \quad ||(L_\delta^n - L_H^n)g||_w \leq \delta(C||g||_s + nD||g||_w). \tag{11}$$

- *Moreover, let us suppose there exists n_1 for which $A\lambda_1^{n_1} < 1$ and $\lambda_2 < 1$ such that*

$$\forall\, v \in \mathcal{S}, ||L_\delta^{n_1}(v)||_w \leq \lambda_2 ||v||_w. \tag{12}$$

Then for any $i \geq 1$

$$\begin{pmatrix} ||L^{in_1}(f)||_s \\ ||L^{in_1}(f)||_w \end{pmatrix} \preceq \begin{pmatrix} A\lambda_1^{n_1} & B \\ \delta C & \delta n_1 D + \lambda_2 \end{pmatrix}^i \begin{pmatrix} ||f||_s \\ ||f||_w \end{pmatrix}.$$

In particular, as before, we have

$$||L^{in_1}g||_s \leq (1/a)\rho^i ||g||_s,$$

and

$$||L^{in_1}g||_w \leq (1/b)\rho^i ||g||_s.$$

The proof of the theorem is essentially the same as the one of Theorem 2. We remark that the main difference between the two propositions is that now we are looking to the behavior of iterates of L_H and L_δ on the whole space $\mathcal{S}$ and not only on the space of zero average measures V.

4. **The Ulam method.** We now give an example of a finite dimensional approximation for the transfer operator which is useful in several cases: the Ulam method.

Let us briefly recall the basic notions. Let us suppose now that X is a manifold with boundary. The space X is discretized by a partition I_δ (with k elements) and the transfer operator L is approximated by a finite rank operator L_δ defined in the following way: let F_δ be the $\sigma-$algebra associated to the partition I_δ, define the projection: π_δ as $\pi_\delta(f) = \mathbf{E}(f|F_\delta)$, then

$$L_\delta(f) := \pi_\delta L \pi_\delta f. \tag{13}$$

In the literature, it is shown from different points of view that taking finer and finer partitions, in suitable systems, the behavior of this finite dimensional approximation converges in some sense to the behavior of the original system, including the convergence of the spectral picture, see, e.g., [11, 5, 10, 20, 3, 7, 3].

We remark that this approximation procedure satisfies the approximation assumption (3) if, for example, Bounded Variation and L^1 norms are considered, see Lemma 14 . This allows the effective use of this discretization in Theorems 2 and 5 for the study of piecewise expanding maps.

5. **Piecewise expanding maps and Lasota Yorke inequalities.** Let us consider a class of maps which are locally expanding but possibly discontinuous at some point. We recall some results, showing that these systems satisfy the assumptions needed in our main theorems, in particular the Lasota Yorke inequality.

Definition 6. A nonsingular function $T : ([0,1], m) \to ([0,1], m)$ is said to be piecewise expanding if

- there is a finite set of points $d_1 = 0, d_2, \ldots, d_n = 1$ such that $T|_{(d_i, d_{i+1})}$ is C^2.
- $\inf_{x \in [0,1]} |D_x T| = \lambda_1^{-1} > 2$ on the set where it is defined.

It is now well known (see, e.g., [19]) that this kind of map has an absolutely continuous invariant measure with bounded variation density.

5.1. **Lasota Yorke inequality.** Let us consider an absolutely continuous measure μ and the following norm:

$$||\mu||_{BV} = \sup_{\phi \in C^1, |\phi|_\infty = 1} |\mu(\phi')|$$

This is related to the classical definition of bounded variation: it is straightforward to see that if μ has density f then $2||f||_\infty \leq ||\mu||_{BV}$.[1]

The following inequality can be established (see, e.g., [12], [21]) for the transfer operator of piecewise expanding maps.

Proposition 7. *If T is piecewise expanding as above and μ is a measure on $[0,1]$*

$$||L\mu||_{BV} \leq \frac{2}{\inf T'} ||\mu||_{BV} + \frac{2}{\min(d_i - d_{i+1})} \mu(1) + 2\mu(|\frac{T''}{(T')^2}|).$$

We remark that, if an inequality of the following form

$$||Lg|| \leq 2\lambda ||g|| + B'|g|_w$$

is established (with $2\lambda < 1$) then, iterating, we have

$$||L^2 g|| \leq 2\lambda ||Lg|| + B'|g|_w = 2\lambda(2\lambda ||Lg|| + B'|g|_w) + B'|g|_w \cdots$$

and thus

$$||L^n g|| \leq 2^n \lambda^n ||Lg|| + \frac{B'}{1 - 2\lambda} |g|_w.$$

5.2. **Piecewise expanding maps with holes.** Let us consider a piecewise expanding map with a hole that is an interval I. Let us see that if the system without hole satisfies a Lasota Yorke inequality and is sufficiently expanding, we can deduce a Lasota Yorke inequality for the system with a hole.

Let indeed consider a starting measure with bounded variation density, f.

If the piecewise expanding map without hole satisfies

$$||Lg||_{BV} \leq 2\lambda ||g||_{BV} + B'||g||_1,$$

then the action of the hole is to multiply Lg by 1_{I^c}, introducing two new jumps which we can bound by $||Lf||_\infty \leq \frac{1}{2}||Lf||_{BV}$, hence

$$||L_H g||_{BV} \leq 2||Lg||_{BV}$$

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[1] If on an interval I, $f \geq q$, then consider a function $\phi \in C^1$ which is -1 on the left of the interval and 1 on the right, and increasing inside. Then $\phi' \geq 0$, $\int \phi' dx = 2$, and $\int f\phi' dx \geq 2q$. We can do similarly if $f \leq -q$, hence $||\mu|| \geq 2||f||_\infty$. The existence of such interval I of course cannot be ensured in general. In this case the use of Lebesgue's density theorem allows us to find an interval where the same argument applies up to a small error.

and the system will satisfy

$$||L_H g||_{BV} \leq 4\lambda ||g||_{BV} + 2B'||g||_1.$$

This is enough to obtain the Lasota Yorke inequality in interesting examples of piecewise expanding maps with holes (see Section 6.3).

We recall that considering the Ulam discretization for these systems, the needed approximation inequality follows again, by Lemma 14, and the assumptions of Section 3 applies.

Remark 8. We remark that no conditions on the size of the hole are needed for our method to be applied.

6. Numerical experiments.

6.1. **Convergence to equilibrium: Lanford map.** In this subsection we estimate the speed of convergence to equilibrium for the map which was investigated in [18]. The map $T : [0,1] \to [0,1]$ is given by

$$T : x \mapsto 2x + \frac{1}{2}x(1-x) \pmod 1.$$

To apply Proposition 7 we have to take into account its second iterate $F := T^2$.

The data below refers to the map F and to the discretization of its transfer operator and are some input and outputs of our algorithm.

$$
\begin{array}{ll}
A = 1 & \lambda_1 \leq 0.32 \\
B \leq 30.6 & \delta = 1/1048576 \\
\lambda_2 < 0.5 & n_1 = 18 \\
C \in [1.94, 1.95] & D \in [70.992, 70.993]
\end{array}
$$

Remark 9. Using the estimates developed in [12], an approximation f_δ for the invariant density f can be computed with an explicit bound on the error. The graph of the map and the computed density are drawn in figure 1a and 1b, respectively. In the plotted case $||f - f_\delta||_1 \leq 0.016$.

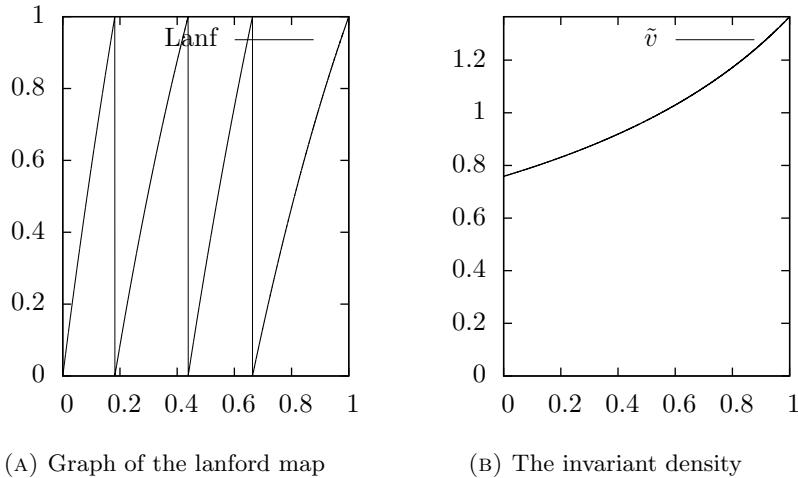(A) Graph of the lanford map   (B) The invariant density

FIGURE 1. The Lanford map

The matrix M that corresponds to our data is such that

$$M \preceq \begin{bmatrix} 1.24 \cdot 10^{-9} & 30.6 \\ 1.86 \cdot 10^{-6} & 0.5013 \end{bmatrix}.$$

Using equation (5) we can compute

$$\rho \in [0.5013, 0.5014].$$

Remark 10. With a simple computation it is possible to see that the coefficients a, b (see Theorem 2) associated to the leading eigenvalue ρ are

$$a = \frac{A\lambda_1^{n_1} - \lambda_2 - \delta n_1 D + \sqrt{(A\lambda_1^{n_1} - \delta n_1 D - \lambda_2)^2 + 4\delta BC}}{A\lambda_1^{n_1} - \lambda_2 - \delta n_1 D + 2B + \sqrt{(A\lambda_1^{n_1} - \delta n_1 D - \lambda_2)^2 + 4\delta BC}}$$

$$b = \frac{2B}{A\lambda_1^{n_1} - \lambda_2 - \delta n_1 D + 2B + \sqrt{(A\lambda_1^{n_1} - \delta n_1 D - \lambda_2)^2 + 4\delta BC}}.$$

In this example,

$$a \in [3.6, 3.7] \cdot 10^{-6}, \quad b \in [0.9999963, 0.9999964].$$

Therefore, if L is the transfer operator associated to $F = T^2$, using (6) and (7), we have the following estimates:

$$\|L^k g\|_{BV} \leq (270839) \cdot (0.5014)^{\lfloor \frac{k}{18} \rfloor} \|g\|_{BV},$$

$$\|L^k g\|_{L^1} \leq (30.7) \cdot (0.5014)^{\lfloor \frac{k}{18} \rfloor} \|g\|_{BV}.$$

We can also use the coefficients of the powers of the matrix (computed using interval arithmetics) to obtain upper bounds as in the following table:

iterations	bound for $\|L^h g\|_{BV}$	bound for $\|L^h g\|_1$
$h = 36$	$5.665 \cdot 10^{-5} \|g\|_{BV} + 15.34 \|g\|_1$	$9.279 \cdot 10^{-7} \|g\|_{BV} + 2.513 \cdot 10^{-1} \|g\|_1$
$h = 72$	$1.424 \cdot 10^{-5} \|g\|_{BV} + 3.855 \|g\|_1$	$2.333 \cdot 10^{-7} \|g\|_{BV} + 6.316 \cdot 10^{-2} \|g\|_1$
$h = 108$	$3.578 \cdot 10^{-6} \|g\|_{BV} + 9.689 \cdot 10^{-1} \|g\|_1$	$5.862 \cdot 10^{-8} \|g\|_{BV} + 1.588 \cdot 10^{-2} \|g\|_1$
$h = 144$	$8.992 \cdot 10^{-7} \|g\|_{BV} + 2.436 \cdot 10^{-1} \|g\|_1$	$1.474 \cdot 10^{-8} \|g\|_{BV} + 3.990 \cdot 10^{-3} \|g\|_1$

6.2. **Convergence to equilibrium: A Lorenz-type map.** In this subsection we estimate the speed of convergence to equilibrium for a Lorenz type 1-dimensional map using the recursive estimate established in section 2. To do so, we use the estimates and the software developed in [12, 14].

The subject of our investigation is the 1 dimensional Lorenz map acting on $I = [0, 1]$ given by

$$T(x) = \begin{cases} \theta \cdot |x - 1/2|^\alpha & 0 \leq x < 1/2 \\ 1 - \theta \cdot |x - 1/2|^\alpha & 1/2 < x \leq 1 \end{cases}$$

with $\alpha = 57/64$ and $\theta = 109/64$.

Please note that since the Lorenz 1-dimensional map does not have bounded derivative, the application of Proposition 7 is not immediate, but the following is true.

Theorem 11 ([14] Theorem 21). *Let T be a 1-dimensional piecewise expanding map, possibly with infinite derivative. Denote by $\{d_i\}$ the set of the discontinuity points of T, increasingly ordered. Fix a parameter $l > 0$ and let*

$$I_l = \left\{ x \in I \mid \left| \frac{T''(x)}{(T'(x))^2} \right| > l \right\}.$$

We have that T satisfies a Lasota Yorke inequality

$$||L\mu||_{BV} \le \lambda_1 ||f||_{BV} + B||f||_1$$

with

$$\lambda_1 \le \frac{1}{2} \int_{I_l} \left| \frac{T''(x)}{(T'(x))^2} \right| dx + \frac{2}{\inf |T'|} \quad B \le \frac{1}{1-\lambda_1} \left(\frac{2}{\min(d_{i+1} - d_i)} + l \right)$$

(and in particular it is possible to choose l in such a way that $\lambda_1 < 1$).

We apply our strategy to the map $F := T^4$. Please remark that, once we have the Lasota-Yorke coefficients and the discretization, the approach is the same as in Subsection 6.1. The data below refers to the map F and to the discretization of its transfer operator and are some input and outputs of our algorithm; to compute the Lasota-Yorke constants we fixed $l = 300$.

$$\begin{array}{ll} A = 1 & \lambda_1 \le 0.884 \\ B \le 4049 & \delta = 1/2097152 \\ \lambda_2 < 0.002 & n_1 = 10 \\ C \in [16.24, 16.25] & D \in [11677.3, 11677.4] \end{array} \tag{14}$$

Remark 12. The graph of the map and its invariant density are drawn in figure 2a and 2b, respectively. Using the estimates developed in [12], if f is the density of the invariant measure and f_δ is the computed density, we have in this case that $||f - f_\delta||_1 \le 0.047$.
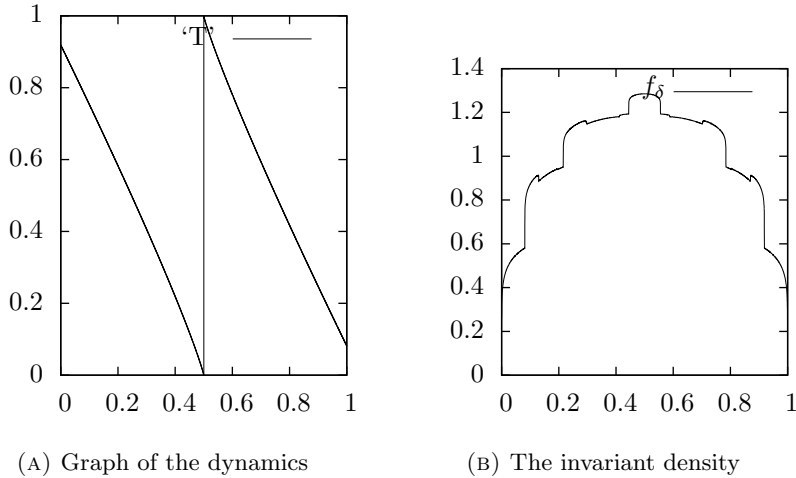

(A) Graph of the dynamics   (B) The invariant density

FIGURE 2. Lorenz 1-dimensional map

The matrix that corresponds to our data is such that

$$M \preceq \left[\begin{smallmatrix} 0.2915 & 4049 \\ 7.75 \cdot 10^{-8} & 0.058 \end{smallmatrix} \right].$$

Remark 13. Please note that the Lorenz map we studied has some features that imply a large coefficient B in the Lasota-Yorke inequality therefore leading to a slower convergence to the equilibrium in the $||.||_{BV}$ norm.

Using equation (5) we can compute

$$\rho \in [0.386, 0.387].$$

Using remark 10 we have that the coefficients a, b are such that

$$a \in ([8.12, 8.13] \cdot 10^{-5}) \quad b \in [0.9999187, 0.9999188].$$

Therefore, if L is the transfer operator associated to $F = T^4$ and using (6) and (7), we have the following estimates:

$$\|L^k g\|_{BV} \leq (16356) \cdot (0.387)^{\lfloor \frac{k}{10} \rfloor} \|g\|_{BV},$$

$$\|L^k g\|_{L^1} \leq (4050) \cdot (0.387)^{\lfloor \frac{k}{10} \rfloor} \|g\|_{BV}.$$

We can also use the coefficients of the powers of the matrix (computed using interval arithmetics) to obtain upper bounds as in the following table:

iterations	bound for $\|L^h g\|_{BV}$	bound for $\|L^h g\|_1$
$h = 20$	$1.163 \cdot 10^{-1} \|g\|_{BV} + 1.414 \cdot 10^3 \|g\|_1$	$2.704 \cdot 10^{-6} \|g\|_{BV} + 3.469 \cdot 10^{-2} \|g\|_1$
$h = 40$	$1.735 \cdot 10^{-2} \|g\|_{BV} + 2.134 \cdot 10^2 \|g\|_1$	$4.082 \cdot 10^{-7} \|g\|_{BV} + 5.025 \cdot 10^{-3} \|g\|_1$
$h = 60$	$2.594 \cdot 10^{-3} \|g\|_{BV} + 31.92 \|g\|_1$	$6.105 \cdot 10^{-8} \|g\|_{BV} + 7.513 \cdot 10^{-4} \|g\|_1$
$h = 80$	$3.880 \cdot 10^{-4} \|g\|_{BV} + 4.774 \|g\|_1$	$9.131 \cdot 10^{-9} \|g\|_{BV} + 1.124 \cdot 10^{-4} \|g\|_1$

6.3. **Escape rates.** In this subsection we estimate the escape rates for a non markov map using the estimates developed in Section 2. We will study

$$T(x) = \frac{23}{5}x \mod 1,$$

with a hole of size $1/8$ centered in $1/2$.

For the system with holes, the Lasota-Yorke inequality has coefficients

$$A = 1 \quad \lambda_1 < 20/23 \quad B' \leq 7.08.$$

Below, some of the data (input and outputs) of our algorithm.

$$
\begin{array}{ll}
A = 1 & \lambda_1 \leq 20/23 \leq 0.87 \\
B \leq 7.08 & \delta = 1/65536 \\
\lambda_2 < 0.5 & n_1 = 17 \\
C \in [14.384, 14.385] & D \in [20.31, 20.32]
\end{array}
$$

The graph of the map and a non rigorous estimation of the conditionally invariant density are drawn in figure 3a and 3b, respectively.

The matrix that corresponds to our data is such that

$$M \preceq \left[\begin{smallmatrix} 0.094 & 7.08 \\ 2.1 \cdot 10^{-4} & 0.506 \end{smallmatrix} \right].$$

Using equation (5) we can compute

$$\rho \in [0.509, 0.5091].$$

Using remark 10 we have that the coefficients a, b are such that

$$a \in ([5.282, 5.283] \cdot 10^{-4}) \quad b \in [0.9994717, 0.9994718].$$

Therefore, if L is the transfer operator associated to T and using (6) and (7), we have the following estimates:

$$\|L^k g\|_{BV} \leq (1900.2) \cdot (0.5091)^{\lfloor \frac{k}{17} \rfloor} \|g\|_{BV},$$

$$\|L^k g\|_{L^1} \leq (7.09) \cdot (0.5091)^{\lfloor \frac{k}{17} \rfloor} \|g\|_{BV}.$$

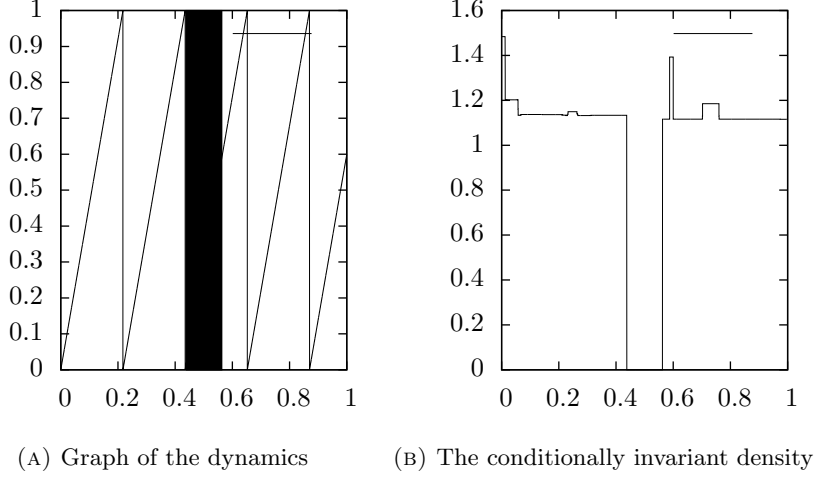(A) Graph of the dynamics   (B) The conditionally invariant density

FIGURE 3. Piecewise linear non-markov map

We can also use the coefficients of the powers of the matrix (computed using interval arithmetics) to obtain upper bounds as in the following table:

iterations	bound for $||L^h g||_{BV}$	bound for $||L^h g||_1$
$h = 34$	$1.034 \cdot 10^{-2}||g||_{BV} + 4.241||g||_1$	$1.315 \cdot 10^{-4}||g||_{BV} + 2.569 \cdot 10^{-1}||g||_1$
$h = 68$	$6.645 \cdot 10^{-4}||g||_{BV} + 1.134||g||_1$	$3.513 \cdot 10^{-5}||g||_{BV} + 6.654 \cdot 10^{-2}||g||_1$
$h = 102$	$1.559 \cdot 10^{-4}||g||_{BV} + 2.939 \cdot 10^{-1}||g||_1$	$9.111 \cdot 10^{-6}||g||_{BV} + 1.724 \cdot 10^{-2}||g||_1$
$h = 136$	$4.025 \cdot 10^{-5}||g||_{BV} + 7.615 \cdot 10^{-2}||g||_1$	$2.361 \cdot 10^{-6}||g||_{BV} + 4.467 \cdot 10^{-3}||g||_1$

Appendix: The approximation inequality from the Lasota Yorke one.
In this section we see that the approximation inequality (3) directly follows from the Lasota Yorke inequality and from natural assumptions on the approximating operator (including the Ulam discretization, which is used in the experiments). Hence the approximation inequality (3) that is assumed in the paper is a natural assumption in a general class of system and approximations.

Lemma 14. *Suppose L satisfies a Lasota Yorke inequality (2) and L_δ is defined by composing with a "projection" π_δ satisfying a certain approximation inequality:*

- $L_\delta = \pi_\delta L \pi_\delta$ *with* $||\pi_\delta v - v||_w \leq \delta ||v||_s$ *for all* $v \in \mathcal{S}$
- π_δ *and L are weak contractions for the norm* $|| \ ||_w$

then $\forall f \in \mathcal{S}$

$$||L^n f - L_\delta^n f||_w \leq \delta \frac{(A\lambda_1 + 1)A}{1 - \lambda_1}||f||_s + \delta B n(A\lambda_1 + 2)||f||_w.$$

Proof. It holds that

$$||(L - L_\delta)f||_w \leq ||\pi_\delta L \pi_\delta f - \pi_\delta L f||_w + ||\pi_\delta L f - L f||_w,$$

but

$$\pi_\delta L \pi_\delta f - \pi_\delta L f = \pi_\delta L(\pi_\delta f - f).$$

Since both π_δ and L are weak contractions and since $||\pi_\delta v - v||_w \leq \delta ||v||_s$,

$$||\pi_\delta L(\pi_\delta f - f)||_w \leq ||\pi_\delta f - f||_w \leq \delta ||f||_s.$$

On the other hand,

$$||\pi_\delta Lf - Lf||_w \leq \delta||Lf||_s \leq \delta(A\lambda_1||f||_s + B||f||_w),$$

which gives

$$||(L - L_\delta)f||_w \leq \delta(A\lambda_1 + 1)||f||_s + \delta B||f||_w. \tag{15}$$

Now let us consider $(L_\delta^n - L^n)f$. It holds that

$$\begin{aligned}
||(L_\delta^n - L^n)f||_w &\leq \sum_{k=1}^{n} ||L_\delta^{n-k}(L_\delta - L)L^{k-1}f||_w \\
&\leq \sum_{k=1}^{n} ||(L_\delta - L)L^{k-1}f||_w \\
&\leq \sum_{k=1}^{n} \delta(A\lambda_1 + 1)||L^{k-1}f||_s + \delta B||L^{k-1}f||_w \\
&\leq \delta \sum_{k=1}^{n} (A\lambda_1 + 1)(A\lambda_1^{k-1}||f||_s + B||f||_w) + B||f||_w \\
&\leq \delta \frac{(A\lambda_1 + 1)A}{1 - \lambda_1}||f||_s + \delta Bn(A\lambda_1 + 2)||f||_w. \qquad \square
\end{aligned}$$

Remark 15. From the above Lemma we have in this case that C and D in (8) are given by

$$C = \frac{(A\lambda_1 + 1)A}{1 - \lambda_1}, \quad D = B \cdot (A\lambda_1 + 2).$$

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