Decay of Correlations, Quantitative Recurrence and Logarithm Law for Contracting Lorenz Attractors

Stefano Galatolo, Isaia Nisoli & Maria Jose Pacifico

Springer

Springer

CrossMark

Decay of Correlations, Quantitative Recurrence and Logarithm Law for Contracting Lorenz Attractors

Stefano Galatolo[1] · **Isaia Nisoli**[2] ·
Maria Jose Pacifico[2]

Abstract In this paper we prove that a class of skew products maps with non uniformly hyperbolic base has exponential decay of correlations. We apply this to obtain a logarithm law for the hitting time associated to a *contracting Lorenz* attractor at all the points having a well defined local dimension, and a quantitative recurrence estimation.

1 Introduction

The term *statistical properties* of a dynamical system $F : M \to M$, where M is a measurable space and F a measurable map, refers to the long time behavior of large sets of trajectories of the system. It is well known that this relates to the properties of the transfer operator, a linear operator associated to the dynamics that embodies how the measures evolve under the action of the system.

✉ Isaia Nisoli
    nisoli@im.ufrj.br

    Stefano Galatolo
    galatolo@dm.unipi.it
    http://users.dma.unipi.it/galatolo/

    Maria Jose Pacifico
    pacifico@im.ufrj.br

1   Dipartimento di Matematica, Largo Bruno Pontecorvo, Pisa, Italy

2   Instituto de Matemática - UFRJ Av. Athos da Silveira Ramos 149, Centro de Tecnologia - Bloco C, Cidade Universitária - Ilha do Fundão, C. P. 68530, CEP 21941-909 Rio de Janeiro, RJ, Brazil

Statistical properties are often a better object to be studied than pointwise behavior. In fact, the future behavior of initial data can be unpredictable, but statistical properties are often regular and their description simpler. Suitable results can be established in many cases, to relate the evolution of measures with that of large sets of points (ergodic theorems, large deviations, central limit, logarithm law, etc...).

In this paper we take the point of view of studying the evolution of measures and its speed of convergence to equilibrium to understand the statistical properties of a class of dynamical systems.

We consider a *Contracting Geometric Lorenz flow* and its perturbations [29], a flow similar to the Geometric Lorenz Flow, but strictly non-uniformly hyperbolic; following the cited paper, we have that if the Contracting Geometric Lorenz Flow admits an attractor and we take a one parameter family of perturbations, parametrized by a parameter a, the there exists an $0 < a_0 < 1$ and $E \subset [0, a_0)$ a set of parameters called the *Rovella parameters*, of positive Lebesgue measure, that admit an attractor (we refer to Sect. 2.3 for a summary of this result).

We prove that the decay of correlations for the map F_a induced on a Poincaré section for a Rovella parameter $a \in E$, is exponential with respect to Lipschitz observables: this permits us to study some of the statistical features of the map and the flow.

We follow the line of [4] exploiting the fact that the system has an invariant contracting foliation and hence can be seen as a skew product whose base transformation has exponential convergence to equilibrium (a measure of how fast iterates of the Lebesgue measure converge to the physical measure), proving the following.

Theorem A *For all Rovella parameters $a \in E$, the two dimensional map F_a associated to the flow has exponential decay of correlations with respect to the physical measure μ_a and Lipschitz observables : there exists $C \geq 0$, $\Lambda \in (0, 1)$ such that*

$$\left| \int f \cdot (g \circ F_a^n) \, d\mu_a - \int g \, d\mu_a \int f \, d\mu_a \right| \leq C\Lambda^n \cdot ||g||_{Lip} \cdot ||f||_{Lip}.$$

The rapid decay of correlations gives, as a consequence, a quantitative recurrence estimation and an estimation for the scaling behavior of the time which is needed to hit small targets (Sect. 7); this kind of result is called a *logarithm law* for the dynamics of the map F_a^n.

A logarithm law is a statement that relates the hitting time to small targets to the local dimension of the physical measure: consider the family of balls $B_r(x_0)$, with center x_0 and radius r, and let us denote the time needed for the orbit of a point x to enter in $B_r(x_0)$ by

$$\tau_r^{F_a}(x, x_0) := \min\{n \in \mathbb{N}^+ : F_a^n(x) \in B_r(x_0)\}.$$

A logarithm law states that as $r \to 0$ the hitting time scales like $1/\mu(B_r)$.

When $x = x_0$ we define

$$\tau_r^{F_a}(x_0) := \tau_r^{F_a}(x_0, x_0),$$

as an indicator for the recurrence time at the point x_0. The exponential decay with respect to Lipschitz observables permits to prove quantitative recurrence statements; as $r \to 0$ the recurrence time also scales like $1/\mu(B_r)$.

To express this more precisely let us consider the local dimensions of a measure μ

$$\overline{d}_\mu(x_0) = \limsup_{r \to 0} \frac{\log \mu(B_r(x_0))}{\log(r)} \quad \text{and} \quad \underline{d}_\mu(x_0) = \liminf_{r \to 0} \frac{\log \mu(B_r(x_0))}{\log(r)} \tag{1.1}$$

representing the scaling rate of the measure of small balls as the radius goes to 0. When the above limits coincide for μ-almost every point, we set $d_\mu = \underline{d}_\mu(x) = \overline{d}_\mu(x)$[1].

---

[1] See Sect. 7 for more details on local dimension and the hitting/return times τ_r.

Theorem B *For the Rovella map F_a, $a \in E$, where E is the set of Rovella parameters, and μ_{F_a} is the invariant SBR measure for the map F_a. For μ_{F_a} almost every x_0 the following quantitative recurrence statement holds:*

$$\liminf_{r \to 0} \frac{\log \tau_r^{F_a}(x_0)}{-\log r} = \underline{d}_{\mu_{F_a}}(x_0), \quad \limsup_{r \to 0} \frac{\log \tau_r^{F_a}(x_0)}{-\log r} = \overline{d}_{\mu_{F_a}}(x_0).$$

Moreover, for each regular point $x_0 \in \Sigma_a$ such that the local dimension of μ_{F_a} at x_0 $d_{\mu_{F_a}}(x_0)$ exists, for the scaling behavior of the hitting time it holds

$$\lim_{r \to 0} \frac{\log \tau_r^{F_a}(x, x_0)}{-\log r} = d_{\mu_{F_a}}(x_0)$$

for μ_{F_a}-almost each $x \in \Sigma_a$.

The logarithm law for the two dimensional map and the integrability of the return time to the Poincaré section with respect to μ_{F_a} imply a logarithm law for the hitting time and recurrence time of the flow, in a way similar to what was done in [11]. Let $x, x_0 \in \Lambda$ and

$$\tau_r^{X_a^t}(x, x_0) = \inf\{t \geq 0 | X_a^t(x) \in B_r(x_0)\} \tag{1.2}$$

be the time needed for the X_a^t-orbit of a point x to enter for the *first time* in a ball $B_r(x_0)$. The number $\tau_r^{X_a^t}(x, x_0)$ is the *hitting time* associated to the flow X_a^t and target $B_r(x_0)$. By this we can also define and consider a quantitative recurrence indicator $\tau_r^{X_a^t}(x_0)$ as done before (see Eq. (7.1) for the definition).

Theorem C *If X_a^t is a contracting Lorenz flow with $a \in E$, where E is the set of Rovella parameters, and μ_{X_a} is the invariant physical measure for the flow X_a^t, then for each regular point $x_0 \in \mathbb{R}^3$ such that $d_{\mu_{X_a}}(x_0)$ exists, for the hitting time behavior it holds*

$$\lim_{r \to 0} \frac{\log \tau_r^{X_a^t}(x, x_0)}{-\log r} = d_{\mu_{X_a}}(x_0) - 1$$

for μ_{X_a}-almost each $x \in \mathbb{R}^3$. While about recurrence time, for μ_{X_a} almost every x_0

$$\liminf_{r \to 0} \frac{\log \tau_r^{X_a}(x_0)}{-\log r} = \underline{d}_{\mu_{X_a}}(x_0) - 1, \quad \limsup_{r \to 0} \frac{\log \tau_r^{X_a}(x_0)}{-\log r} = \overline{d}_{\mu_{X_a}}(x_0) - 1.$$

This extends the results of [11] to the contracting Geometric Lorenz flow and its perturbations.

1.1 Organization of the Text

This paper is organized as follows. In Sect. 2 we introduce the main object of this article, the contracting Lorenz flow. In Sect. 3 we make explicit the main properties of the one dimensional map associated to a contracting Lorenz flow. In Sect. 4 we recall some general results on convergence and correlation decay for skew-products with contracting fibers from [4], that will be used to prove Theorem A. In Sect. 5 we show how to extend some results about decay of correlations and convergence to equilibrium for Hölder observables to generalized bounded variation observables. In Sect. 6 we establish exponential decay of correlations with respect to Lipschitz observables for the two dimensional map associated to a contracting Lorenz flow and prove Theorem A. In Sect. 7 we show some consequences of the decay of
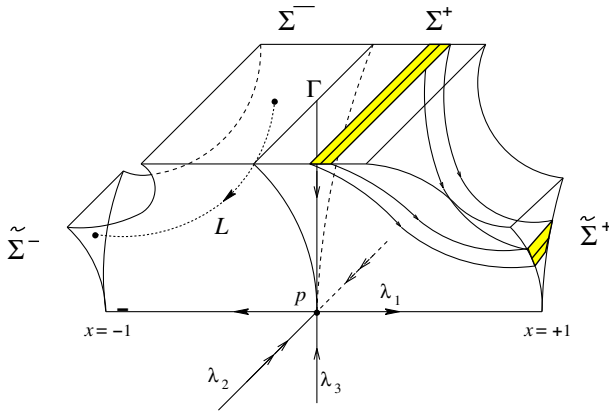
Fig. 1 Behaviour near the origin

correlations proved above as hitting time and quantitative recurrence estimations, proving Theorem B and C. Finally, in Sect. 1 we explain a result about linearization and properties of the Poincaré map associated to a contracting Lorenz flow needed along the paper.

2 Contracting Lorenz Flows

In this section we present the family of dynamical systems studied at [29], which are the object of our paper. The starting point for its definition is the geometric contracting Lorenz flow, a system which is constructed similarly to the classical geometric Lorenz flow, in which the uniformly expanding direction is replaced by a strict nonuniformly expanding direction (for a discussion of this terminology, we refer to the introduction of [21])

We describe informally this construction following [26]. Let $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$ be a linear vector field in the cube $[-1, 1]^3$, with a singularity at the origin $(0, 0, 0)$. Suppose the eigenvalues λ_i, $1 \leq i \leq 3$ satisfy the relations

$$- \lambda_2 > -\lambda_3 > \lambda_1 > 0, \quad r = -\frac{\lambda_2}{\lambda_1}, \quad s = -\frac{\lambda_3}{\lambda_1}, \quad r > s + 3. \tag{2.1}$$

It is worth to remark that $\lambda_1 + \lambda_3 < 0$ in the contracting case while in the definition of the usual geometric Lorenz flow the condition is $\lambda_1 + \lambda_3 > 0$. The condition $r > s + 3$ is used in [29] to guarantee the existence of a C^3 uniformly contracting stable foliation for the Poincaré first return map of perturbations of the geometric contracting Lorenz flow.

Let $\Sigma^- = \{(x, y, 1) \mid -1/2 \leq x \leq 0, |y| \leq 1/2\}$, $\Sigma^+ = \{(x, y, 1) \mid 0 \leq x \leq 1/2, |y| \leq 1/2\}$ and $\Sigma = \Sigma^+ \cup \Sigma^-$. In Fig. 1 we can see the behaviour of the field near the origin. With some computations, it is possible to see that the flow reaches the transverse section $x = 1$ (a similar reasoning works for $x = -1$) obeying the following law:

$$\tilde{F}(x, y, 1) = (1, yx^r, x^s) \tag{2.2}$$

Outside the cube, as in the Geometric Lorenz case, to imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for our flow, we let the flow return to the cross section Σ through a flow described by a suitable composition of a rotation $R_\pm$, an expansion $E_{\pm\rho}$ and a translation $T_\pm$. The resulting effect of the flow outside the cube when we arrive on Σ may be represented by a rotation and the expansion which have the form:

$$R_\pm(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}, \quad E_{\pm,\rho}(x, y, z) = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

ρ is such that $\rho \cdot (1/2)^s \leq 1$. This condition is an hypothesis on the behavior of the vector field outside a neighborhood of the origin, and insures that the image of the map is contained in Σ. Further, we can also take ρ sufficiently small to guarantee that the contraction along the stable foliation is stronger than ρ. The condition on the eigenvalues expressed in Eq. (2.1) gives the necessary condition to obtain this contraction, see [29, Remarks, p. 240].

The translations $T_\pm$ are chosen in such a way that the unstable direction starting from the origin is sent to the boundary of Σ and the images of $\tilde{\Sigma}_\pm$ are disjoint.

It is possible to find a flow X_0^t that realizes this construction, as it is described in [11,26].

Composing the expression in (2.2) with $R_\pm$, $E_{\pm\rho}$ and $T_\pm$ we write an explicit formula for $F_0 : \Sigma \to \Sigma$, the Poincaré first return map of the geometric contracting Lorenz flow on the section Σ:

$$F_0(x, y) = (T_0(x), G_0(x, y)) \tag{2.3}$$

$$T_0(x) = \begin{cases} -\rho|x|^s + 1/2 & x > 0 \\ \rho|x|^s - 1/2 & x < 0 \end{cases}, G_0(x, y) = \begin{cases} y|x|^r + c_0 & x > 0 \\ -y|x|^r + c_1 & x < 0 \end{cases},$$

where c_0, c_1 are real numbers depending on the choice of the translations $T_\pm$, r and s are as in (2.1) and $\rho \leq (1/2)^{-s}$.

2.1 Properties of the One-Dimensional Map T_0

It is proved in [29, Item 4, p. 240] that the map T_0 (an example of which is represented in Fig. 2) satisfies the following properties:

(a) T_0 is onto and piecewise C^3, with two branches. The order[2] of $T_0'(x) = O(x^{s-1})$ at $x = 0$ where $s = -\frac{\lambda_3}{\lambda_1}$ and $s - 1 > 0$,
(b) T_0 has a discontinuity at $x = 0$, $T_0(0^+) = 1/2$, $T_0(0^-) = -1/2$,
(c) $T_0'(x) < 0$ for every $x \neq 0$,
(d) $\max_{x>0} T_0'(x) = T_0'(1/2)$ and $\max_{x<0} T_0'(x) = T_0'(-1/2)$.

There exist $\rho \leq (1/2)^{-s}$ such that also the following hypothesis are satisfied:

(e) The points $1/2$ and $-1/2$ are preperiodic repelling for T_0.
(f) The map T_0 has negative schwarzian derivative.

Under these hypothesis, in [29] it is proved that X_0 *has an attractor* Λ_0.

Remark 2.1 If $\rho = (1/2)^{-s}$, then the map T_0 is topologically conjugated to the doubling map and it is called, in the literature, a "full Rovella map" [26]. For a "full Rovella map", the existence of an attractor and of an a.c.i.m. for T_0 are easily proved.

2.2 Summary of the Construction of the Perturbations of X_0^t

To ease reading, we will summarize some of the results regarding the perturbations of the flow X_0^t. This construction is going to be made more explicit in the next subsections.

Now suppose X_0^t is the flow induced by X_0 constructed above and X^t is the flow of a C^3-small perturbation X of X_0. Then X^t has a hyperbolic singular point $\mathcal{O}$ near the origin

---

[2] We say that $f(x)$ is $O(g(x))$ at $x = x_0$ if there exists M, δ such that $|f(x)| \leq M|g(x)|$ when $0 < |x - x_0| < \delta$.

Fig. 2 A 1-dimensional contracting Lorenz map
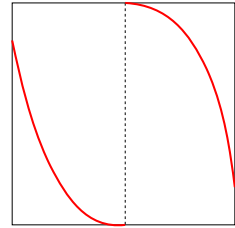
with eigenvalues near those of X_0^t. The fact that X^t is also conjugate to its linear part via a C^1-conjugacy [31] is true if there are no resonances between the eigenvalues. Since this is an open, dense condition, we may assume it. Moreover the conjugacy is only valid near $\mathcal{O}$. Since away from the singular point, only a finite amount of time is involved, no problem arises. Thus Σ is still a cross section to X and, as proved in [29, Proposition, p. 241], we may assume there exists a C^3 open neighborhood $\mathcal{U}$ of X_0 such that the flow of each $X \in \mathcal{U}$ leaves invariant a C^3 nearly vertical foliation $\mathcal{F}_X$ of Σ that varies continuously with X. Thus for all $X \in \mathcal{U}$ the induced first return map $F_X : \Sigma \to \Sigma$ leaves invariant a C^3 nearly vertical foliation $\mathcal{F}_X$ of Σ implying that this first return map can be written as

$$F_X(x, y) = (T_X(x), G_X(x, y)).$$

2.3 Λ_0 is 2-Dimensionally Almost Persistent

Unlike the case of expanding Lorenz flow, not all $X \in \mathcal{U}$, the open C^3-neighborhood of X_0 above, presents an attractor, but the existence of the attractor is persisten in a measure theoretical sense. This was proved in [29]; in this section we will introduce the necessary definitons and cite the main result.

Let us recall the definition of an attractor and stability in the measure theoretical sense.

Definition 1 Let X be a vector field, with associated flow X^t. A set Λ is an attractor for X if it is compact, invariant under X^t, transitive (i.e., contains a dense orbit) and it has a compact neighborhood U, called *local basin* of Λ, such that $\Lambda = \bigcap_{t \geq 0} X^t(U)$.

Definition 2 Given a subset S of a finite dimensional Riemannian manifold $\mathcal{M}$, we say that x is a density point of S, if, denoting by m the Lebesgue measure on $\mathcal{M}$, $B_r(x)$ the ball of radius r and centered at x, we have:

$$\lim_{r \to 0} \frac{m(B_r(x) \cap S)}{m(B_r(x))} = 1.$$

Definition 3 Given a subset S of a Banach space $\mathcal{B}$, we say that $x \in S$ is a point of k-dimensional full density of S if there exists a C^∞ submanifold $\mathcal{N} \subset \mathcal{B}$, containing x and having codimension k, such that for every k- dimensional manifold $\mathcal{M}$ intersecting $\mathcal{N}$ transversally at x, then x is a full density point of $S \cap \mathcal{M}$ in $\mathcal{M}$.

Definition 4 An attractor Λ of a flow $X^t \in C^\infty$ is k-dimensionally almost persistent if it has a local basin U such that X is a k-dimensional full density point of the set of flows $Y^t \in C^\infty$ for which $\Lambda_Y = \cap_{t > 0} Y^t(U)$ is an attractor.

In [29] it is proved that the attractor Λ_0 constructed as above is 2-*dimensionally almost persistent in the C^3 topology*, see item (b) of Theorem at p. 235.

The proof of this result is similar to the proof of [5] where it is proved that, for the map $f_a(x) = 1 - ax^2$ there exists a set of parameters of positive Lebesgue measure for which f_a has an absolutely continuous invariant measure. The main step in the proof is to exploit further the hypotheses on the initial vector field X_0 that lead to the expression at Eq. (2.3) for its Poincaré map. This allows to reduce the problem to the analysis of the one dimensional map induced by the flow X_0^t. A central result is the following, which proves the persistence of a C^3 stable foliation.

Theorem 1 [29, Proposition, p. 241] *There exists an open neighborhood $\mathcal{U}$ of X_0 such that the flow of each $X \in \mathcal{U}$ admits a C^3 stable one dimensional foliation in U that varies continuously with X.*

By Frobenius Theorem, for each $X \in \mathcal{U}$ we can find a transversal section to X^t (and depending from X^t), near Σ and consisting of pieces of leafs of the stable foliation.

After a change of coordinates, for each $X \in \mathcal{U}$ the first return map associated to X can be written as

$$F_X(x, y) = (T_X(x), G_X(x, y)).$$

The one dimensional map T_X induced by F_X through the foliation is C^3 in $x \neq 0$, 0 is the discontinuity and the critical point. Furthermore, we can choose coordinates on the transversal section such that $T_X(0^+) = -1/2$, $T_X(0^-) = 1/2$.

Let $\mathcal{U}$ be a C^3 neighbourhood of X_0 as in Theorem 1 and define $\mathcal{N}$ as

$$\mathcal{N} = \{Y \in \mathcal{U} \mid \exists k^+, k^- \text{ so that } T_Y^{k^+}(1/2), T_Y^{k^-}(-1/2) \text{ are periodic repelling}\}.$$

Note that if $\mathcal{U}$ is small enough then $\mathcal{N}$ is a codimension 2 submanifold containing X_0.

We can now cite the main theorem proved in [29]:

Theorem 2 ([29]) *Let $\mathcal{M}$ be a 2-dimensional C^3-submanifold of $\mathcal{U}$ intersecting $\mathcal{N}$ transversally, at X_0. Let $\{X_a\}$ be a one parameter family of vector fields, contained in $\mathcal{M}$, such that the functions $a \mapsto T_{X_a}(\mp 1/2)$ have derivative 1 at $a = 0$. Then there is a subset $E \subset (0, a_0)$ of parameters called the* Rovella parameters*, with a_0 close to 0 and 0 a full density point of E such that*

$$\lim_{a \to 0} \frac{|E \cap (0, a)|}{a} = 1, \text{ with } \Lambda_{X_a} = \cap_{t \geq 0} X_a^t(U) \text{ an attractor.}$$

This implies that Λ_0 is 2-dimensionally almost persistent.

To ease notation, if $\{X_a\}$ is a one parameter family of vector fields as above, we denote by

$$F_a(x, y) = (T_a(x), G_a(x, y))$$

the Poincaré map associated to X_a.

In the remaining of the paper, we restrict ourselves to the setting when $\{X_a\}$ is one of those one parameter families. We will denote the eigenvalues of X_a at the singularity by $\lambda_{1,a}, \lambda_{2,a}, \lambda_{3,a}$.

Lemma 2.2 ([29]) *For all Rovella parameter $a \in E$ the induced 1-dimensional map T_a satisfies the following additional properties:*

(C1) $T_a'(x) = O(x^{s(a)-1})$ *as $x \to 0$ where $s(a) = -\lambda_{3,a}/\lambda_{1,a}$.*

(C2) *there is $\lambda_c > 1$ such that for all $a \in E$, the points $1/2$ and $-1/2$ have Lyapunov exponents greater that λ_c:*

$$(T_a^n)'(\pm 1/2) > \lambda_c^n, \quad \text{for all} \quad n \geq 1;$$

(C3) *there is $\alpha > 0$ such that for all $a \in E$ the basic assumption holds:*

$$|T_a^{n-1}(\pm 1/2)| > e^{-\alpha n}, \quad \text{for all} \quad n \geq 1;$$

(C4) *the forward orbits of the points $\pm 1/2$ under T_a are dense in $[-1/2, 1/2]$ for all $a \in E$.*

(C5) *for all $a \in E$, T_a has negative schwarzian derivative.*

Remark 2.3 Item C1 depends on the possibility of linearizing the vector field X_a near the origin; while this hypothesis is not explicit in [29], this is the reason why we assume X_a to be a C^3 one parameter family in the space of C^∞ vector fields, i.e., a C^3 map $\Xi : \mathbb{R} \supset (-\varepsilon, \varepsilon) \to C^\infty(\mathbb{R}^3)$, such that $\Xi(0) = X_0$.

We refer to Sect. 1 for a discussion about the order of the derivatives of T_a and of G_a.

The hypothesis of having a C^k family in the space of C^r vector fields is widely used in the contracting Lorenz setting [23,25].

These properties have strong consequences on the statistical properties of the one dimensional T_a, as we will relate in Sect. 3. In our work we show how the statistical properties of T_a imply some statistical properties for the flow X_a^t associated to a Rovella parameter $a \in E$.

Remark 2.4 The following properties of X_a for all $a \in [0, a_0]$, used in our paper, follow from the properties of the linearization near the origin. We refer to Sect. 1 for some explicit computations; let $r(a) = -\lambda_{2,a}/\lambda_{1,a}$:

(1) $\frac{\partial G_a}{\partial y}(x, y) = O(x^{r(a)})$ as x goes to 0, with $r(a) > s(a)+3$ which implies that $r(a) > 3$,

(2) the map G_a is contracting along the leaves of the stable foliation, due to the fact that $\lambda_{2,a}$ is near $\lambda_2 < 0$,

(3) the order of $\frac{\partial G_a}{\partial y}(x, y) = O(x^{l(a)})$ and $l(a) \geq \min(s(a) - 1, r(a), r(a) - 1)$, i.e., $l(a) \geq s(a) - 1 > 0$

(4) if $\log(x)$ is integrable with respect to the invariant measure μ_a of T_a then the first return time of X_a^t to Σ is integrable.

3 Further Properties of the One Dimensional Contracting Lorenz Map

In [22], conditions (C1) and (C3) were used to prove the existence of an ergodic absolutely continuous invariant probability measure for Rovella parameters. In order to obtain uniqueness of that measure, Metzger needed to consider a slightly smaller class of parameters (still with full density at 0), for which conditions (C2) and (C3) imply a strong mixing property. But, in [2], Alves and Soufi deduce the uniqueness of the ergodic absolutely continuous invariant probability measure for $a \in E$ not assuming any strong mixing property. Hence, for each $a \in E$, the map T_a has a unique SRB measure μ_a.

We now recall some recent results of [2] on the statistical properties of the contracting Lorenz one dimensional maps that we will use in our paper.

To state these statistical properties, we start recalling some definitions and facts about T_a, with a a Rovella parameter.

Definition 5 We say that T_a is **non-uniformly expanding** if there is a $c > 0$ such that for Lebesgue almost every $x \in I$

$$\lim_{n \to \infty} \inf \frac{1}{n} \sum_{i=0}^{n-1} \log(T'_a(T_a^i(x))) > c. \tag{3.1}$$

Definition 6 We say that T_a has **slow recurrence to the critical set** if for every $\epsilon > 0$ there exists $\delta > 0$ such that for Lebesgue almost every $x \in I$ it holds

$$\lim_{n \to \infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(T_a^i(x), 0) \leq \epsilon, \tag{3.2}$$

where d_δ is the δ-truncated distance, defined as

$$d_\delta(x, y) = \begin{cases} |x - y|, & \text{if } |x - y| \leq \delta, \\ 1, & \text{if } |x - y| > \delta. \end{cases} \tag{3.3}$$

Definition 7 The **expansion time function** is defined as

$$\mathcal{E}_a(x) = \min\{N \geq 1; \frac{1}{n} \sum_{i=0}^{n-1} \log T'_a(T_a^i(x)) > c, \forall n \geq N\},$$

which is well defined and finite almost everywhere in I, provided (3.1) holds almost everywhere.

Fixing $\epsilon > 0$ and choosing $\delta > 0$ conveniently, we define the **recurrence time function**

$$\mathcal{R}_a(x) = \min\{N \geq 1; \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(T_a^i(x), 0) < \epsilon, \forall n \geq N\},$$

which is defined and finite almost every where in I, as long as (3.2) holds almost everywhere.

Definition 8 We define the **tail set at time** n to be the set of points which at time n have not yet achieved either the uniform exponential growth of the derivative or the uniform slow recurrence:

$$\Gamma_a^n = \{x \in I; \mathcal{E}_a(x) > n \text{ or } \mathcal{R}_a(x) > n\}.$$

Theorem 3 ([2, Theorem A]) *Each T_a, with $a \in E$, is non-uniformly expanding and has slow recurrence to the critical set. Moreover, there are $C > 0$ and $\tau > 0$ such that for all $a \in E$ and $n \in \mathbb{N}$, it holds*

$$m(\Gamma_a^n) \leq Ce^{-\tau n}, \tag{3.4}$$

where m is the Lebesgue measure on I.

In [2] the authors deduced several interesting consequences from (3.4), which follow from Theorem 2 and Theorem 3 of the seminal paper [33]. Their results involve the class of Hölder continuos functions[3] with a given exponent $\alpha > 0$, denoted by $H(\alpha)$.

The main result in [2] is that for all Rovella parameter $a \in E$:

---

[3] We will denote by

$$\text{Höl}_\alpha(f) = \sup_{x, y \in \mathbb{I}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

and by $||f||_{H(\alpha)} := ||f||_\infty + \text{Höl}_\alpha(f)$ the α-Hölder norm.

(1) T_a has a unique ergodic absolutely continuous invariant probability measure μ_a;
(2) the measure μ_a has exponential decay of correlations for $H(\alpha)$-observables against $L^\infty(\mu_a)$ observables.

A concept strictly related to the decay of correlations is the concept of speed of convergence to equilibrium.

Definition 3.1 We say that (F, μ) has exponential convergence to equilibrium with respect to norms $\|.\|_a$ and $\|.\|_b$, if there are $C \in \mathbb{R}^+$, $\Lambda \in (0, 1)$ such that for $f \in L^1(m)$, $g \in L^1(\mu)$ and for all $n \geq 1$ it holds

$$Conv_n(f, g) := \left| \int f \cdot (g \circ T^n) \, dm - \int g \, d\mu \int f \, dm \right| \leq C\Lambda^n \cdot \|g\|_a \cdot \|f\|_b.$$

From (3.4) and [33] it follows that the systems we consider have exponential convergence to equilibrium with respect to L^∞ and Hölder observables (see also [7, Appendix B] for a standard procedure to get a uniform statement for all the observables in the given classes).

Proposition 3.2 *Let $a \in E$, let μ_a be the absolutely continuous invariant measure of T_a. Then (T_a, μ_a) has exponential convergence to equilibrium in the following sense. There are $C \in \mathbb{R}^+$, $\Lambda \in (0, 1)$ such that for each $f \in H(\alpha)$ and $g \in L^\infty(\mu_a)$*

$$Conv_n(f, g) \leq C\Lambda^n \cdot \|g\|_\infty \cdot ||f||_{H(\alpha)}$$

In the next section we will see how a result on the convergence to equilibrium for a certain map can be extended to a skew product with contracting fibers, based on that map.

4 General Result on Convergence and Correlation Decay for Skew-Products with Contracting Fibers

We recall some general results from [4]. For this, let $\mathbb{I} = [-1/2, 1/2]$ the unit interval and denote by $\mathbb{Q} = \mathbb{I} \times \mathbb{I}$. We consider maps $F : \mathbb{Q} \to \mathbb{Q}$ preserving a regular foliation, which contracts the leaves and whose quotient map (the induced map on the space of leaves) has exponential convergence to equilibrium. In this section we will prove that also the map F has exponential convergence to equilibrium.

Definition 4.1 If $f : \mathbb{Q} \to \mathbb{Q}$ is integrable, we denote by $\pi(f) : \mathbb{I} \to \mathbb{I}$ the function $\pi(f) : x \mapsto \int_{\mathbb{I}} f(x, t) \, dt$.

Let us consider the following anisotropic norm, considering Lipschitz regularity only on the vertical, y, direction. Let $\| \cdot \|_{y-\text{Lip}}$ be defined by

$$\|g\|_{y-\text{Lip}} = \|g\|_{sup} + \text{Lip}_y(g), \tag{4.1}$$

where

$$\|g\|_{sup} := \sup_{x,y \in \mathbb{I}} |g(x, y)| \quad \text{and} \quad \text{Lip}_y(g) := \sup_{\substack{x, y_1, y_2 \in \mathbb{I} \\ y_1 \neq y_2}} \frac{|g(x, y_2) - g(x, y_1)|}{|y_2 - y_1|}. \tag{4.2}$$

The following is proved in [4, Theorem 1]

Theorem 4.2 *Let $F : \mathbb{Q} \circlearrowleft$ be a Borel function such that $F(x, y) = (T(x), G(x, y))$. Let μ be an F-invariant measure with absolutely continuous marginal μ_T on the x-axis which, moreover, is T-invariant. Let us suppose that*

(1) *(T, μ_T) has exponential convergence to equilibrium with respect to the norm $\|\cdot\|_\infty$ (the L^∞ norm) and to a norm (on the base space) which we denote by $\|\cdot\|_{Base}$,*
(2) *suppose $\|\cdot\|_{Base}$ is stronger than $\|\cdot\|_\infty$, i.e., $\|\cdot\|_{Base} \geq \|\cdot\|_\infty$,*
(3) *T is nonsingular with respect to the Lesbegue measure, piecewise continuous and monotonic: there is a collection of intervals $\{I_i\}_{i=1,...,m}$, $\cup I_i = I$ such that on each I_i, T is an homeomorphism onto its image,*
(4) *F is a contraction on each vertical leaf: G is λ -Lipschitz in y with $\lambda < 1$.*

Then (F, μ) has exponential convergence to equilibrium in the following sense. There are $C \in \mathbb{R}^+$, $\Lambda \in (0, 1)$ such that

$$Conv_n(f, g) \leq C\Lambda^n \cdot \|g\|_{y-Lip} \cdot (\|\pi(f)\|_{Base} + \|f\|_1)$$

for each $f \geq 0$.

Now let us relate convergence to equilibrium to decay of correlations. This will be done by the following statement (see [4, Theorems 2 and 3])

Theorem 4.3 *Let $F : \mathbb{Q} \circlearrowleft$ be a Borel function such that $F(x, y) = (T(x), G(x, y))$, μ an F-invariant probability measure with absolutely continuous T-invariant marginal μ_T on the x-axis and satisfying*

(1) *(T, μ_T) has exponential convergence to equilibrium with respect to the norm $\|\cdot\|_\infty$ and to a norm $\|\cdot\|_{Base}$;*
(2) *T is nonsingular with respect to the Lesbegue measure, piecewise continuous and monotonic: there is a collection of intervals $\{I_i\}_{i=1,...,m}$, $\cup I_i = \mathbb{I}$ such that on each I_i, T is an homeomorphism onto its image.*
(3) *F is a uniform contraction on each vertical leaf.*
(4) *Moreover, let us assume that that there are C_1, $K \in \mathbb{R}$ and a seminorm $\|\cdot\|_\square$ such that*

$$\|\pi(f \circ F^n)\|_{Base} + \|f \circ F^n\|_\square \leq C_1 K^n (\|\pi(f)\|_{Base} + \|f\|_{y-Lip} + \|f\|_\square), \quad \forall n \geq 1.$$

Then F has exponential decay of correlations: there are $C_2 > 0$, $\Lambda \in (0, 1)$ such that

$$\left| \int f \cdot (g \circ F^n) \, d\mu - \int g \, d\mu \int f \, d\mu \right| \leq C_2 \Lambda^n \|g\|_{y-Lip} (\|f\|_{y-Lip} + \|\pi(f)\|_{Base} + \|f\|_\square)$$

for all $f, g : \mathbb{Q} \to \mathbb{R}$ and $n \geq 0$.

The notations $\|\cdot\|_{Base}$ and $\|\cdot\|_\square$ emphatize that these are respectively a norm for functions on the base space of the skew product, and seminorm for functions on the whole space, the square $\mathbb{Q}$. In the next section we will find concrete examples of a norm and a seminorm with the required properties (the p-variation norm in the next section and the seminorm $Var^\square$ in the following one).

5 Decay of Correlations for Adapted Function Spaces

Many results about decay of correlations and convergence to equilibrium are obtained for Hölder observables. Yet, in systems with discontinuities, this class of functions is not the most natural, since it is not preserved by the transfer operator. We show how to extend those results to generalized bounded variation observables. This extension is necessary to apply Theorem 4.3.

5.1 Functions of Bounded *p*-Variation

We recall the main definitions and basic results about bounded *p*-Variation (see [19]).

Given a function we define its universal *p*-Variation as the following adaptation of the usual notion of bounded variation.

Definition 5.1 Let $g : [0, 1] \to \mathbb{R}$ and let:

$$\mathrm{Var}_p(g, x_1, \ldots, x_n) = \left(\sum_{i \leq n} |g(x_i) - g(x_{i+1})|^p \right)^{\frac{1}{p}}.$$

The **universal *p*-Variation** is:

$$\mathrm{Var}_p(g) = \sup_{\mathcal{P}} \mathrm{Var}_p(g, x_1, \ldots, x_n),$$

where $\mathcal{P}$ is the collection of all the finite subdivisions of $[0, 1]$. Let

$$\mathrm{UBV}_p = \{g : \mathrm{Var}_p(g) < \infty\}$$

be the space of functions of bounded universal *p*-Variation.

In the following m be the Lebesgue measure on the unit interval, $\epsilon > 0$ and $h : [0, 1] \to \mathbb{C}$. We define

$$\mathrm{osc}(h, \epsilon, x) = \mathrm{ess\,sup}\{|h(y_1) - h(y_2)| : y_1, y_2 \in B_\epsilon(x)\},$$

where $B_\epsilon(x)$ is the ball centered in x with radius ϵ , and the essential supremum is taken with respect to the Lebesgue measure. Now let us define

$$\mathrm{osc}_p(h, \epsilon) = \|\mathrm{osc}(h, \epsilon, x)\|_p, \quad 1 \leq p \leq \infty,$$

where the *p*-norm is taken with respect to m.

Remark 5.2 $\mathrm{osc}_p(h, *) : (0, A] \to [0, \infty]$ is a non decreasing function and $\mathrm{osc}_p(h, \epsilon) \geq \mathrm{osc}_1(h, \epsilon)$.

Fixed $0 \leq r \leq 1$, set $R_{p,r,n} = \{h | \forall \epsilon \in (0, A], \mathrm{osc}_p(h, \epsilon) \leq n\epsilon^r\}$ and $S_{p,r} = \cup_{n \in \mathbb{N}} R_{p,r,n}$.

We can now define:

(1) $BV_{p,r}$ as the space of *m*-equivalence classes of functions in $S_{p,r}$
(2) $\mathrm{Var}_{p,r}(h) = \sup_{0 < \epsilon \leq A}(\epsilon^{-r} \mathrm{osc}_p(h, \epsilon))$ (we remark that this definition depends on a fixed constant A and that $\mathrm{Var}_{p,r}(h) \geq \mathrm{Var}_{1,r}(h)$).
(3) for $h \in BV_{p,r}$ we define $\|h\|_{p,r} := \mathrm{Var}_{p,r}(h) + \|h\|_p$.

It turns out that $BV_{p,r}$ with the norm $\|h\|_{p,r}$ is a Banach space; see [19, Thm. 1.13]. In the following we will fix $A = 1$.

Proposition 5.3 $UBV_p \subseteq BV_{p,\frac{1}{p}} \subseteq BV_{1,\frac{1}{p}}$ *for all* $1 \leq p < \infty$. *Moreover*

$$Var_{1,\frac{1}{p}}(h) \leq Var_{p,\frac{1}{p}}(h) \leq 2^{\frac{1}{p}} Var_p(h). \tag{5.1}$$

In what follows we need to compare the $\| \cdot \|_{p,r}$ norm with the $L^\infty(m)$ norm. The following Lemma will be useful (see [4, Lemma 2]).

Lemma 5.4 *If $f \in BV_{1,r}$ ($r \leq 1$), then $f \in L^\infty(m)$ and*

$$\|f\|_\infty \leq A^{r-1} \cdot \|f\|_{1,r},$$

where A is the constant in the definition of $\|\cdot\|_{1,r}$ (see item 2 above).

5.2 Hölder Convergence to Equilibrium Implies Convergence to Equilibrium for Bounded p-Variation

Suppose we have a system having exponential convergence to equilibrium with $H(\alpha)$ and L^∞ observables, let us estimate the convergence for $f \in BV_{1,p}$ and $g \in L^\infty$.

Proposition 5.5 *If for each $f \in H(\alpha)$ and $g \in L^\infty$ we have convergence to equilibrium with speed Φ :*

$$Conv_n(f,g) := \left| \int g \circ T^n \, f \, dm - \int g \, d\mu \int f \, dm \right| \leq \|g\|_\infty \|f\|_{H(\alpha)} \Phi(n),$$

then for each f, g respectively in $BV_{1,\alpha}$ and L^∞ it holds

$$Conv_n(f,g) \leq \|g\|_\infty \|f\|_{1,\alpha} 6\sqrt{\Phi(n)}.$$

Proof Let us consider $\rho_\epsilon = \frac{1}{2\epsilon} 1_{B(0,\epsilon)}$ a multiple of the characteristic function of an interval of radius ϵ, small.

Let us consider $f \in BV_{1,\alpha}$, approximate f with $f_\epsilon = f * \rho_\epsilon$ (the convolution with ρ_ϵ) and estimate the integral:

$$\left| \int g \circ T^n \, f \, dm \right| \leq \left| \int g \circ T^n \, (f + f_\epsilon - f_\epsilon) \, dm - \int g \, d\mu \int f + f_\epsilon - f_\epsilon \, dm \right|$$

$$\leq \int \left| g \circ T^n \, (f - f_\epsilon) \right| dm + \int g \, d\mu \int |f - f_\epsilon| \, dm$$

$$+ \left| \int g \circ T^n \, f_\epsilon \, dm - \int g \, d\mu \int f_\epsilon \, dm \right|$$

$$\leq 2\|g\|_\infty \|f - f_\epsilon\|_1 + \|g\|_\infty \|f_\epsilon\|_{H(\alpha)} \Phi(n).$$

Now let us estimate $\|f - f_\epsilon\|_1$

$$\|f - f * \rho_\epsilon\|_1 \leq \int_I \left| \int_{B(0,\epsilon)} [f(x-y) - f(x)] \rho_\epsilon(y) \, dy \right| dx$$

$$\leq \int_I \left| \sup_{y \in B(x,\epsilon)} [f(x-y) - f(x)] \right| dx$$

$$\leq osc_1(f, \epsilon).$$

We bound the Hölder seminorm of f_ϵ

$$\text{Höl}_\alpha(f * \rho_\epsilon(x)) = \sup_{x_1, x_2 \in I} |x_1 - x_2|^{-\alpha} \left| \int_{B(0,\epsilon)} [f(x_1 - y) - f(x_2 - y)] \rho_\epsilon(y) dy \right|,$$

by the definition of ρ_ϵ

$$\left| \int_{B(0,\epsilon)} [f(x_1 - y) - f(x_2 - y)] \rho_\epsilon(y) dy \right| \leq \begin{cases} 2\epsilon^{-1} |x_1 - x_2| \, \|f\|_\infty \text{ if } |x_1 - x_2| \leq \epsilon \\ 2\|f\|_\infty \text{ if } |x_1 - x_2| > \epsilon \end{cases}.$$

Hence

$$\text{Höl}_\alpha(f * \rho_\epsilon(x)) \leq \sup_{x_1,x_2 \in I} \left(\left\{ \begin{array}{l} |x_1 - x_2|^{-\alpha+1} 2\epsilon^{-1} ||f||_\infty \text{ if } |x_1 - x_2| \leq \epsilon \\ |x_1 - x_2|^{-\alpha} 2 ||f||_\infty \text{ if } |x_1 - x_2| \geq \epsilon. \end{array} \right\} \right).$$

It follows

$$\text{Höl}_\alpha(f * \rho_\epsilon(x)) \leq 2\epsilon^{-\alpha} ||f||_\infty \leq 2\epsilon^{-\alpha} ||f||_{1,\alpha}.$$

Summarizing

$$|\int g \circ T^n f \, dm - \int g \, d\mu \int f \, dm|$$

$$\leq 2||g||_\infty ||f - f_\epsilon||_1 + ||g||_\infty ||f_\epsilon||_{H(\alpha)} \Phi(n)$$

$$\leq ||g||_\infty (2\text{osc}_1(f, \epsilon) + 4\epsilon^{-\alpha} ||f||_{1,\alpha} \Phi(n))$$

$$\leq ||g||_\infty (2\epsilon^\alpha ||f||_{1,\alpha} + 4\epsilon^{-\alpha} ||f||_{1,\alpha} \Phi(n))$$

$$= ||g||_\infty ||f||_{1,\alpha} (2\epsilon^\alpha + 4\epsilon^{-\alpha} \Phi(n)).$$

For each n we can take ϵ such that $\epsilon^{2\alpha} = \Phi(n)$ we have $\epsilon^\alpha = \sqrt{\Phi(n)}$ and

$$\left| \int g \circ T^n f \, dm - \int g \, d\mu \int f \, dm \right| \leq 6||g||_\infty ||f||_{1,\alpha} \sqrt{\Phi(n)}.$$

$\square$

By this proposition it follows that if a system has exponential convergence to equilibrium with respect to Hölder observables, it has also exponential convergence with respect to generalized bounded variation ones.

6 Decay of Correlations for the Two Dimensional Contracting Lorenz Map: Proof of Theorem A

In the following section we explain which are the norms involved, in our case, in the statements of Sect. 4 and we will check that the 2-dimensional Rovella maps satisfies the hypothesis of our theorem, implying the proof of Theorem A.

As showed in Proposition 3.2 the one-dimensional Rovella map for a Rovella parameter $a \in E$ satisfies exponential convergence to equilibrium with respect to Hölder and L^∞ observables. By using the results in Sect. 5 and in particular by Proposition 5.5 we have the following.

Corollary 6.1 *If a is a Rovella parameter, the one dimensional Rovella map satisfies exponential convergence to equilibrium with respect to the norms $||.||_{1,\alpha}$ and $||.||_\infty$.*

We will need another definition of variation for maps with two variables. Similarly to the one dimensional case, if $f : \mathbb{Q} \to \mathbb{R}$ and $x_i \leq x_2 \leq \cdots \leq x_n$, let us define the variation on the square of f as

$$\text{Var}^\square(f, x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{1 \leq i \leq n} |f(x_i, y_i) - f(x_{i+1}, y_i)|.$$

We then consider the supremum $\mathrm{Var}^{\square}(f, x_1, \ldots, x_n, y_1, \ldots, y_n)$ over all subdivisions x_i and all choices of the y_i

$$\mathrm{Var}^{\square}(f) = \sup_n \left(\sup_{(x_i \leq x_2 \leq \cdots \leq x_n) \in \mathcal{I}, (y_i) \in \mathcal{I}} \mathrm{Var}^{\square}(f, x_1, \ldots, x_n, y_1, \ldots, y_n) \right).$$

We want to apply Theorem 4.3 using $|| \cdot ||_{1,\alpha}$ as the norm $|| \cdot ||_{Base}$ and $\mathrm{Var}^{\square}(\cdot)$ as the seminorm $|| \cdot ||_{\square}$. Thus we need to prove that item 4 of Theorem 4.3 is satisfied for this seminorm; to do so, thanks to [4, Lemma 16] the only thing we need to prove now is the following.

Lemma 6.2 *For each $a \in [0, a_0]$ we have that $\mathrm{Var}^{\square}(G_a) < \infty$*

Proof Remembering the definition:

$$\mathrm{Var}^{\square}(G_a, x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_n |G_a(x_i, y_i) - G_a(x_{i+1}, y_i)|.$$

By Remark 2.4 we have that $\frac{\partial G_a}{\partial x}$ is bounded for all $x \in I$ and there exists M such that

$$\left| \frac{\partial G_a}{\partial x} \right| \leq M |x|^{s(a)-1},$$

with $s(a) - 1 > 0$. Moreover, we can observe that, if $x < 0$ and $\tilde{x} > 0$ we have that

$$|G(x, y) - G(\tilde{x}, y)| < 1.$$

Therefore, if $-1/2 = x_1 < \cdots < x_k < 0 < x_{k+1} < \cdots < x_n = 1/2$ then

$$\sum_0^{k-1} |G(x_i, y_i) - G(x_{i+1}, y_i)| + |G(x_k, y_i) - G(x_{k+1}, y_i)|$$

$$+ \sum_{k+1}^{n-1} |G(x_i, y_i) - G(x_{i+1}, y_i)| \leq$$

$$\sum_0^{k-1} |\frac{\partial G}{\partial x}(\xi_i)| |x_{i+1} - x_i| + 1 + \sum_{k+1}^{n-1} |\frac{\partial G}{\partial x}(\xi_i)| |x_{i+1} - x_i| \leq 1 + M,$$

where $\xi_i \in [x_i, x_{i+1}]$ for $i \neq k$. $\qquad\square$

From this, as in [4], by Theorems 4.2 and 4.3 follows the decay of correlation with respect to Lipschitz observables, proving Theorem A.

Proposition 6.3 *If a is a Rovella parameter, the two dimensional Rovella map F_a has exponential convergence to equilibrium and exponential decay of correlations: There are $C \in \mathbb{R}^+$, $\Lambda \in (0, 1)$ such that for $n \geq 1$:*

$$Conv_n(f, g) \leq C\Lambda^n \cdot ||g||_{y-Lip} \cdot (||\pi(f)||_{1,\alpha} + ||f||_1)$$

$$\left| \int f \cdot (g \circ F^n) \, d\mu - \int g \, d\mu \int f \, d\mu \right| \leq C\Lambda^n \cdot ||g||_{y-Lip} (||f||_{y-Lip} + ||\pi(f)||_{1,\alpha} + \mathrm{Var}^{\square} f).$$

7 Quantitative Recurrence and Logarithm Law, Proof of Theorem B and C

In this section we will show some consequences of the decay of correlations proved above. We show how these results imply hitting time and quantitative recurrence estimations.

Let us consider a discrete time dynamical system (X, T, μ), where (X, d) is a metric space and $T : X \to X$ is a measurable map preserving a finite measure μ. Let us consider two points x, y in X and the time which is necessary for the orbit of x to approach y at a distance less than r

$$\tau_r(x, y) = \min\{n \in \mathbb{N}^+ : d(T^n(x), y) < r\};$$

if $x = y$ we will use the notation $\tau_r(x) := \tau_r(x, x)$.

We consider the behavior of $\tau_r(x, y)$ as $r \to 0$. In many interesting cases this is a power law $\tau_r(x, y) \sim r^R$. When $x \neq y$ the exponent is a quantitative measure of how fast the orbit starting from x approaches a point y. When $x = y$ the exponent R gives a quantitative measure of the speed of recurrence of an orbit near to its starting point, and this will be a quantitative recurrence indicator.

Definition 9 Let ν be a measure on X, metric space. The upper and lower local dimension are defined as:

$$\overline{d}_\nu(x) = \limsup_{r \to 0} \frac{\log(\mu(B_r(x)))}{\log(r)} \quad \underline{d}_\nu(x) = \liminf_{r \to 0} \frac{\log(\mu(B_r(x)))}{\log(r)}.$$

If $\overline{d}_\nu(x) = \underline{d}_\nu(x)$ we say the local dimension exists at x and denote it by $d_\nu(x) = \underline{d}_\nu(x)$; if the local dimension exists and $d_\nu(x)$ is constant for ν a.e. x, we say the system is exact dimensional.

The results of [28] and [8], give a quantitative estimation for these indicators for rapidly mixing systems, and can be summarized in the following theorem (see also [12]) that directly implies Theorem B.

Theorem 7.1 ([8,12,28]) *Let (X, T, ν) be a measure preserving system with a decay of correlations with respect to Lipschitz observables faster than any polynomial rate. Let x, $y \in X$,*

(1) *if the local dimension $d_\nu(y)$ exists then for ν-almost every x:*

$$\limsup_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = \liminf_{r \to 0} \frac{\log(\tau_r(x, y))}{-\log(r)} = d_\nu(y)$$

(2) *If $X \subseteq \mathbb{R}^d$ for some d, then for ν-almost every x such that $\underline{d}_\nu(x) > 0$:*

$$\limsup_{r \to 0} \frac{\log(\tau_r(x))}{-\log(r)} = \overline{d}_\nu(x) \quad and \quad \liminf_{r \to 0} \frac{\log(\tau_r(x))}{-\log(r)} = \underline{d}_\nu(x).$$

These general result can be applied to prove Theorem B.

Proof of Theorem B Observe that for a Rovella two dimensional map F_a, $a \in E$ a Rovella parameter, the SRB measure μ_{F_a} has exponential decay of correlations with respect to Lipschitz observables and that $\underline{d}_{\mu_{F_a}}(x) > 0$ almost everywhere (the projection of the measure on the basis is absolutely continuous). Applying Theorem 7.1 we have Theorem B. □

Now we extend the result to the contracting Lorenz flow, following [10,11]; we need to check that the return time to the section is integrable.

Proposition 7.2 *Let $a \in E$ be a Rovella parameter, and let μ_a be the invariant measure for the one dimensional map T_a. Then:*

$$\int_{-1/2}^{1/2} -\log|x| d\mu_a \leq \infty.$$

Proof In [2] it was proved that for each Rovella parameter the map T_a has slow recurrence to the critical set (Definition 6) we will use this to give a bound on the integral of $\log|x|$.

First of all, let $I_\delta := (-\delta, \delta)$ and let $J_\delta = I \setminus I_\delta$:

$$\int_I -\log|x| d\mu_a = \int_{J_\delta} -\log|x| d\mu_a + \int_{I_\delta} -\log|x| d\mu_a;$$

first we will bound

$$\int_{J_\delta} -\log|x| d\mu_a \leq \int_{J_\delta} -\log(\delta) d\mu_a \leq \int_I -\log(\delta) d\mu_a = -\log(\delta).$$

Now we observe that, if we denote by χ_{I_δ} the characteristic function of I_δ and by d_δ the truncated distance as in Definition 6:

$$-\log|x| \cdot \chi_{I_\delta} = -\log(d_\delta(x, 0)),$$

thus

$$\int_{I_\delta} -\log|x| d\mu_a = \int_I -\log|x| \cdot \chi_{I_\delta} d\mu_a = \int_I -\log(d_\delta(x, 0)) d\mu_a.$$

Let now $\phi_k(x) = \min\{-\log(d_\delta(x, 0)), k\}$; trivially $||\phi_k||_\infty \leq k$, therefore $\phi_k \in L^1(\mu_a)$ and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_k(T_a^i(x)) = \int_I \phi_k d\mu_a,$$

for μ_a a.e. x. Moreover, for all x, k and n:

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi_k(T_a^i(x)) \leq \frac{1}{n} \sum_{i=0}^{n-1} -\log(d_\delta(T_a^i(x), 0)).$$

We argue by contradiction; suppose $\int_I -\log(d_\delta(x, 0)) d\mu_a = +\infty$; therefore

$$+\infty = \lim_{k \to +\infty} \int_I \phi_k(x) d\mu_a = \lim_{k \to +\infty} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_k(T_a^i(x)) \quad \mu_a - a.e.$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} -\log(d_\delta(T_a^i(x), 0)) \leq \epsilon < +\infty$$

since the slow recurrence to the critical set holds for a set of full Lebesgue measure. □

Remark 7.3 In particular, since the return time on the section for the Contracting Geometric Lorenz Flow is controlled by $-\log|x|/\lambda_1$ this implies that the return time on the section for the Contracting Geometric Lorenz Flow is integrable.

Remark 7.4 Proposition 7.2 is needed in the Rovella case because the density of μ_a may be unbounded near $x = 0$; this does not happen in the Lorenz case since we have stronger results concerning the regularity of the density of the invariant measure of the one dimensional map that cannot be applied in the non uniformly hiperbolic case.

We show now that the return time on the section is integrable for all the Rovella parameters.

Proposition 7.5 *If the neighborhood $\mathcal{U}$ is small enough, then the return time to the section is integrable for each Rovella parameter $a \in E$.*

Proof Due to the linearization argument in Sect. 1, we know that there exist two constants C_1 and C_2 such that the return time to the cross section τ_X satisfies:

$$C_1 - \frac{\log |x|}{\tilde{\lambda}_1} \le \tau_X(x, y) \le C_2 - \frac{\log |x|}{\tilde{\lambda}_1}.$$

Since $-\log |x|$ is integrable for each Rovella parameter, we have the thesis. $\qquad \square$

Consider a flow X^t in $\mathbb{R}^3$ having a transversal section Σ whose first return time is integrable. As before let $F : \Sigma \setminus \Gamma \to \Sigma$ be the first return map associated. Let μ_F be an ergodic invariant measure for F. Now, if we consider a flow having such a map as its Poincaré section and integrable return time, we can construct an SRB invariant measure μ_X for the flow. Let $x, x_0 \in \mathbb{R}^3$ and

$$\tau_r^{X^t}(x, x_0) = \inf\{t \ge 0 | X^t(x) \in B_r(x_0)\}$$

be the time needed for the X-orbit of a point x to enter for the *first time* in a ball $B_r(x_0)$. The number $\tau_r^{X^t}(x, x_0)$ is the *hitting time associated to* the flow X^t and $B_r(x_0)$.

If the orbit X^t starts at x_0 itself let us consider the return time in the ball and denote

$$\tau_r^X(x_0) = \inf\{t \in \mathbb{R}^+ : X^t(x_0) \in B_r(x_0), \exists i < t, s.t. X^t(x_0) \notin B_r(x_0)\}. \quad (7.1)$$

If $x, x_0 \in \Sigma$ and $B_r^\Sigma(x_0) = B_r(x_0) \cap \Sigma$ we denote

$$\tau_r^\Sigma(x, x_0) = \min\{n \in N^+; F^n(x) \in B_r^\Sigma(x_0)\}$$

the *hitting time associated to* the discrete system F.

Given any x we recall that we denote with $t(x)$ the first strictly positive time, such that $X^{t(x)}(x) \in \Sigma$ (the *return time of x to Σ*). A relation between $\tau_r{}^X(x, x_0)$ and $\tau_r^\Sigma(x, x_0)$ is proved in [11] (Proposition 5.2). Let $x \in \mathbb{R}^3$ and $\pi(x)$ be the projection on Σ given by $\pi(x) = y$ if x is on the orbit of $y \in \Sigma$ and the orbit from y to x does not cross Σ (if $x \in \Sigma$ then $\pi(x) = x$).

Proposition 7.6 *Under the assumptions listed above, there is a full μ_X measure set $B \subset \mathbb{R}^3$ such that if $x_0 \in \mathbb{R}^3$ is regular and $x \in B$ it holds (provided the second limits exist)*

$$\lim_{r \to 0} \frac{\log \tau_r^{X^t}(x, x_0)}{-\log r} = \lim_{r \to 0} \frac{\log \tau_r^\Sigma(\pi(x), \pi(x_0))}{-\log r}. \quad (7.2)$$

By Theorem B, applying this last proposition to the 2-dimensional system (Σ, F_a, μ_a) and the contracting geometric Lorenz flow and its invariant SRB measure, we get the logarithm law and the quantitative recurrence statement for the flow, i.e., Theorem C.

Appendix: Linearization and Properties of the Poincaré Map

To study the order of partial derivative of the first return map to the transverse section we use a C^1 linearization near the singularity; we will use Theorem 7.1 p. 257 of [15] in its version for flows.

Theorem 8.1 *Let $n \in \mathbb{Z}^+$ be given. Then there exists an integer $N = N(n) \geq 2$ such that: if Γ is a real non-singular $d \times d$ matrix with eigenvalues $\gamma_1, \ldots, \gamma_d$ satisfying*

$$\sum_{i=1}^{d} m_i \gamma_i \neq \gamma_k \quad \text{for all } k = 1, \ldots, d \text{ and } 2 \leq \sum_{j=1}^{d} m_i \leq N(n)$$

and if $\dot{\xi} = \Gamma(\xi) + \Xi(\xi)$ and $\dot{\zeta} = \Gamma \zeta$, where $\xi, \zeta \in \mathbb{R}^d$ and Ξ is of class C^N for small $||\xi||$ with $\Xi(0) = 0$, $\partial_\xi \Xi(0) = 0$; then there exists a C^n diffeomorphism R from a neighborhood of $\xi = 0$ to a neighborhood of $\zeta = 0$ such that $R\xi(t)R^{-1} = \zeta(t)$ for all $t \in \mathbb{R}$ and initial conditions for which the flows $\zeta(t)$ and $\xi(t)$ are defined in the corresponding neighborhood of the origin.

Since the resonance conditions are open, there exists an $N = N(1)$ such that, if we choose the eigenvalues of the geometric contracting Lorenz Flow respecting the resonance conditions, there exists a C^N neighborhood of X_0 such that all the vector fields in the neighborhood respect the resonance conditions and can be C^1-linearized. Generically, the linear part of a vector field $\tilde{X}$ in such a neighborhood is different from the linear part of X_0; our aim is not to find a common linearization for all the fields in the neighborhood but to ensure the fact that $\tilde{X}$ is C^1-linearizable.

Behaviour Near the Fixed Point

Let $\tilde{X}$ be in a C^N neighborhood of X_0 such that the resonance condition are still satisfied; then $\tilde{X}$ can be C^1-linearized.

Denote by $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3$ the eigenvalues of $\tilde{X}$ at the fixed point, and denote by $\tilde{r} = -\tilde{\lambda}_2/\tilde{\lambda}_1$ and by $\tilde{s} = -\tilde{\lambda}_3/\tilde{\lambda}_1$.

First, we will study the behaviour of the flow in a neighborhood of the singularity and then use the information about the existence of a foliated atlas to obtain informations on the order of derivatives for the first return Poincaré maps.

Near the singularity, there exists a coordinate system such that the singularity p is in 0, the field is given by $\tilde{X} = (\tilde{\lambda}_1 x, \tilde{\lambda}_2 y, \tilde{\lambda}_3 x)$ and there exists sections $\tilde{\Sigma} = \{z = \varepsilon, |x| \leq \varepsilon, |y| \leq \varepsilon\}$, $\tilde{\Sigma}_+ = \{x = +\varepsilon, \}$ and $\tilde{\Sigma}_- = \{x = -\varepsilon\}$.

By the same computations as for the geometric contracting Lorenz flow X_0 we have that the map from $[-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \subset \tilde{\Sigma}$ to $\tilde{\Sigma}_+$ is given by:

$$\tilde{F}(x, y, 1) = (1, \tilde{G}(x, y), \tilde{T}(x)) = (\varepsilon, \varepsilon^{-\tilde{r}} y \cdot x^{\tilde{r}}, \varepsilon^{1-\tilde{s}} x^{\tilde{s}}),$$

and that the time taken between $\tilde{\Sigma}$ and $\tilde{\Sigma}_+$ is given by

$$\tau(x, y, 1) = \frac{\log(\varepsilon) - \log(|x|)}{\tilde{\lambda}_1}.$$

Remark that both arguments work also for $\tilde{\Sigma}_-$.

We can choose the neighborhood such that $\tilde{s} - 1 > 0$ and $\tilde{r} - \tilde{s} > 3$. Therefore, as x approaches 0 we have that

$$\frac{\partial \tilde{T}}{\partial x} = O(x^{\tilde{s}-1}), \quad \frac{\partial \tilde{G}}{\partial x} = O(x^{\tilde{r}-1}), \quad \frac{\partial \tilde{G}}{\partial y} = O(x^{\tilde{r}}),$$

with $\tilde{s} > 1, \tilde{r} > 1$.

Behaviour Far from the Fixed Point

If $N \geq 3$ we know from [29] that all the vector fields in a neighborhood of X_0 preserve a stable foliation. Let $\tilde{X}$ be a vector field in a C^N neighborhood of X_0 such that the resonance conditions are preserved, and whose flow preserves a stable foliation. Let $F_{\tilde{X}}(x, y) = (T_{\tilde{X}}(x), G_{\tilde{X}}(x, y))$ be the first return map of the flow to the section Σ; we will study the behaviour of its partial derivatives as x approaches 0.

With an abuse of notation, we will denote by $\tilde{\Sigma}$, $\tilde{\Sigma}_+$ and $\tilde{\Sigma}_-$ the preimages under the linearizing change of coordinates of the sections $\tilde{\Sigma}$, $\tilde{\Sigma}_+$ and $\tilde{\Sigma}_-$; the important property is that these preimages are locally C^1-manifolds and that they are made up of pieces of stable leaves. In particular, we can see $\tilde{\Sigma}_+$ as a graph of a function of (y, z), where the constant leaves are given by constant z.

We want to show that, since the flow preserves the stable foliation, then, recalling the results from Sect. 1 we have that

(1) the order of $\frac{\partial T}{\partial x}$ as x goes to 0 is the same as the order of $\frac{\partial \tilde{T}}{\partial x}$, i.e., $\tilde{s} - 1$
(2) the order of $\frac{\partial G}{\partial y}$ is the same as the order of $\frac{\partial \tilde{G}}{\partial y}$, i.e., $\tilde{r}$
(3) the order of $\frac{\partial G}{\partial x}$ is at least the minimum of the orders of $\frac{\partial \tilde{T}}{\partial x}, \frac{\partial \tilde{G}}{\partial x}, \frac{\partial \tilde{G}}{\partial y}$, i.e., at least $\tilde{s} - 1$.
(4) if $\log(x)$ is integrable with respect to the invariant measure, the return time to Σ is integrable.

Since the flow of $\tilde{X}$ has no singularities besides p, the map $\varphi_2(y, z) = (\mu(z), \nu(y, z))$ that takes $\tilde{\Sigma}_+$ into Σ is a diffeomorphism that sends lines $z = \text{const}$ into lines $x = \text{const}$ and the map $\varphi_1(x, y) = (\chi(x), \zeta(x, y))$ that takes Σ into $\tilde{\Sigma}$ is a map that send lines $x = \text{const}$ into lines $x = \text{const}$.

By direct computation we have that

$$DF_{\tilde{X}} = D\varphi_2 \circ D\tilde{F} \circ D\varphi_1$$

$$= \begin{bmatrix} \frac{\partial \mu}{\partial z} \frac{\partial \tilde{T}}{\partial x} \frac{\partial \chi}{\partial x} & 0 \\ \frac{\partial \nu}{\partial y} \left(\frac{\partial \tilde{G}}{\partial x} \frac{\partial \chi}{\partial x} + \frac{\partial \tilde{G}}{\partial y} \frac{\partial \zeta}{\partial y} \right) + \frac{\partial \nu}{\partial z} \frac{\partial \tilde{T}}{\partial x} \frac{\partial \chi}{\partial x} & \frac{\partial \nu}{\partial y} \frac{\partial \tilde{G}}{\partial y} \frac{\partial \zeta}{\partial y} \end{bmatrix}.$$

Since φ_2 is a diffeomorphism, therefore $\frac{\partial \nu}{\partial y} \cdot \frac{\partial \mu}{\partial z} \neq 0$, and since φ_1 is a diffeomorphism $\frac{\partial \chi}{\partial x} \cdot \frac{\partial \zeta}{\partial y} \neq 0$. This proves item (1), (2), (3).

Since the flow has no singularities both the time between Σ and $\tilde{\Sigma}$ and the time between $\tilde{\Sigma}_+$ and Σ are bounded; this implies item (4).

References

1. Afraimovich, V.S., Bykov, V.V., Shil'nikov, L.P.: On the appearence and structure of the Lorenz attractor. Dokl. Acad. Sci. USSR **234**, 336–339 (1977)

2. Alves, J.F., Soufi, M.: Statistical stability and limit laws for Rovella maps. Nonlinearity **25**(12), 3527–3552 (2012)
3. Araújo, V., Pacifico. M.J.: Three-dimensional flows, volume 53 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Heidelberg, . With a foreword by Marcelo Viana (2010)
4. Araujo, V., Galatolo, S., Pacifico, M.J.: Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors. Math. Z. **276**(3–4), 1001–1048 (2014)
5. Benedicks, M., Carleson, L.: On iterations of $1 - ax^2$ on $(-1, 1)$. Ann. Math. **122**, 1–25 (1985)
6. Benedicks, M., Carleson, L.: The dynamics of the Hénon map. Ann. Math. **133**, 73–169 (1991)
7. Collet, P., Chazottes, J.R., Schmitt, B.: Statistical consequences of Devroye inequality for processes. Applications to a class of non-uniformly hyperbolic dynamical systems. Nonlinearity **18**, 2341–2364 (2005)
8. Galatolo, S.: Dimension and hitting time in rapidly mixing systems. Math. Res. Lett. **14**(5), 797–805 (2007)
9. Galatolo, S.: Hitting time in regular sets and logarithm law for rapidly mixing dynamical systems. Proc. Am. Math. Soc. **138**(7), 2477–2487 (2010)
10. Galatolo, S., Nisoli, I.: Shrinking targets in fast mixing flows and the geodesic flow on negatively curved manifolds. Nonlinearity **24**, 3099–3113 (2011)
11. Galatolo, S., Pacifico, M.J.: Lorenz like flows: exponential decay of correlations for the poincaré map, logarithm law, quantitative recurrence. Ergod. Theory Dyn. Syst. **30**, 703–1737 (2010)
12. Galatolo, S., Rousseau, J., Saussol, B.: Skew products, quantitative recurrence, shrinking targets and decay of correlations. Ergod. Theory Dyn. Syst. **35**(06), 1814–1845 (2015)
13. Gouezel, S.: Decay of correlations for nonuniformly expanding systems Bull. Soc. Math. de France **134**, 1–31 (2006)
14. Guckenheimer, J., Williams, R.F.: Structural stability of Lorenz attractors. Publ. Math. IHES **50**, 59–72 (1979)
15. Hartmann, P.: Ordinary Differential Equations, Classics in Applied Mathematics (Volume 38), SIAM, 2nd edn (2002)
16. Hirsch, M., Pugh, C., Shub, M.: Invariant manifolds. Lect, vol. 583. Notes in Math. Springer, New York (1977)
17. Holland, M., Melbourne, I.: Central limit theorems and invariance principles for Lorenz attractors. J. Lond. Math. Soc. (2) **76**(2), 345–364 (2007)
18. Ionescu-Tulcea, C.T., Marinescu, G.: Théorie ergodique pour des classes d'operations non complètement continues. Ann. Math. **52**, 140–147 (1950)
19. Keller, G.: Generalized bounded variation and applications to piecewise monotonic transformations. Z. Wahrsch. Verw. Gebiete **69**(3), 461–478 (1985)
20. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
21. Luzzatto, S.: Stochastic-like behaviour in non-uniformly expanding maps in A. Katok, B. Hasselblat, Handbook of Dynamical Systems Vol. 1A. Elsevier (2005)
22. Metzger, R.J.: Sinai Ruelle Bowen measures for contracting Lorenz maps and flows. Ann. Inst. Henri Poincaré **17**, 247–276 (2000)
23. Morales, C.A., Pacifico, M.J., Pujals, E.R.: Strange attractors across the boundary of hyperbolic systems. Commun. Math. Phys. **211**, 527–558 (2000)
24. Morales, C.A., Pacifico, M.J., Pujals, E.R.: Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. Ann. Math. (2) **160**(2), 375–432 (2004)
25. Morales, C.A., Pacifico, M.J., San, B.: Martin. Contracting Lorenz attractors through resonant double homoclinic loops. SIAM J. Math. Anal. **38**(1), 309–332 (2006)
26. Pacifico, M.J., Todd, M.: Thermodynamic formalism for contracting Lorenz flows. J. Stat. Phys. **139**(1), 159–176 (2010)
27. Pugh, C., Shub, M., Wilkinson, A.: Hölder foliations. Duke Math. J. **86**, 517–546 (1997)
28. Rousseau, J., Saussol, B.: Poincaré recurrence for observations. Trans. A.M.S. **362**(11), 5845–5859 (2006)
29. Rovella, A.: The dynamics of perturbations of the contracting Lorenz attractor. Bull. Brazil. Math. Soc **24**, 233–259 (1993)
30. Steinberger, T.: Local dimension of ergodic measures for two-dimensional lorenz transformations. Ergod. Theory Dyn. Syst. **20**(3), 911–923 (2000)
31. Takens, F.: Partially hyperbolic fixed points. Topology **10**, 133–147 (1971)
32. Young, L.S.: Statistical properties of dynamical systems with some hyperbolicity. Ann. Math. **147**, 585–650 (1998)
33. Young, L.S.: Recurrence times and rates of mixing. Israel J. Math. **110**, 153–188 (1999)