LECTURE NOTES ON KÄHLER GEOMETRY

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Abstract. We collect notes from the mini-course on Kähler geometry that was held at the Universidade de São Paulo during late 2011.

This set of notes originated from the desire of the author to present a brief summary of a certain topic in Kähler geometry, that being of the geometry of extremal Kähler metrics, as a mini-course at the USP. This is a very active and interesting area of current research. In part this can be thought of as the natural candidate for a distinguished class of metric that is admitted by a *typical* compact complex manifold of Kähler type. The author is by no means an expert in this field, but is well aware of the beauty of this field and desired to learn more by leading some lectures. It quickly became apparent that less ambitious goals should be set, with the eventual result being the content of these notes, as presented over five lectures. This has been broken down into seven sections, listed as lectures, but this is largely for reasons of coherence and clarity.

It should be clear that no claim to originality is made in any way here. The principal reference that was used was a set of semi-formal notes prepared by Paul Gauduchon on extremal metrics. Additionally, the standard texts by Griffiths and Harris, and Huybrechts were also used. The derivation of the equation defining the Eguchi-Hanson metric comes from the short article of Donaldson. One feature that is perhaps of value in this note is that I have not seen as complete and explicit derivation of the Eguchi-Hanson metric, from the ALE point of view, elsewhere. In this note we derive and solve the equation, say why the resulting metric is complete at infinity, and show that that it extends smoothly across 0.

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1. Lecture 1

1.1. **Linear Algebra.** Let M be a smooth manifold. An *almost complex structure* on M is an endomorphism of the tangent bundle

$$\begin{aligned} J : T_M &\to T_M \\ J^2 &= -Id \end{aligned}$$

If we complexify the tangent bundle we can extend J to be a complex linear endomorphism of $T_M \otimes \mathbb{C}$. $J^2 = -1$ so the complexification splits as a direct sum of $+i$ and $-i$-eigenbundles of J

$$T_M \otimes \mathbb{C} = T_M^{1,0} \oplus T_M^{0,1}.$$

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The inclusion of T_M into $T_M \otimes \mathbb{C}$, followed by the projections onto the two factors defines *real*-linear isomorphisms

$$\begin{array}{rcl} T_M & \to & \mathcal{T}_M^{1,0} \\ X & \mapsto & X^{1,0} = \frac{1}{2}(X - iJX), \end{array}$$

with the $(0,1)$-vector defined similarly, with a $+$-sign.

A complex valued covector on M, $\alpha : T_M \to \mathbb{C}$, is said to be of type $(1,0)$ if $\alpha(T^{1,0}) = 0$, and of type $(0,1)$ if it annihilates $T^{1,0}$. This in particular means that α is of type $(1,0)$ if and only if

$$\alpha(JX) = i\alpha(X)$$

for all $X \in T_M$.

We can extend the endomorphism J to act on the exterior algebras of T_M and T_M^* by defining

$$\begin{array}{rcl} J : \Lambda^p T_M^* & \to & \Lambda^p T_M^* \\ (J \cdot \alpha)(X_1, \ldots, X_k) & = & \alpha(J^{-1}X_1, \ldots, J^{-1}X_k) \\ & = & (-1)^k \alpha(X_1, \ldots X_k). \end{array}$$

This definition is made so that J is well-behaved with respect to contractions of vectors into differential forms. This is demonstrated in the following obvious facts.

(1) $J = Id$ on $\Lambda^0 T_M^*$.
(2) $\alpha(X) = (J\alpha)(JX)$.
(3) The forms of type $(1,0)$, which we denote by $\Lambda_M^{1,0}$, are the eigenvectors of J of value $-i$.
(4) The complex 2-forms decompose as $\Lambda^2 \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}$. The first two summands together form the $+1$-eigenspace and the third term forms comprises the -1-eigenspace.
(5) If $R \in T_M \otimes T_M^*$ is an endomorphism, $(JR)(v) = JR(J^{-1}v)$. This means that R is J-invariant, or a $+1$-eigenvector for J, if and only if R commutes with J.

1.2. **Nijenhuis Tensor.**

Definition 1.1. The *Nijenhuis tensor* of an almost-complex manifold (M, J) is given by

$$\begin{array}{rcl} N_J : T_M & \times & T_M \to T_M \\ N_J(X,Y) & = & \frac{1}{4}\left([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]\right). \end{array}$$

This leads us to the celebrated theorem of Newlander and Nirenberg on the existence of complex analytic charts.

Theorem 1.2. $N_J = 0$ *if and only if there exists an atlas of charts with holomorphic transition maps such that* J *is induced by multiplication by* i *on vectors in* $\mathbb{C}^n$ *in any such local chart.*

In this case, if $N_J = 0$, we say that the almost complex structure J is *integrable*.

We can observe that the Nijenhuis tensor can be expressed in terms of Lie derivative,

$$N_J(X,Y) = \frac{1}{4}\left((\mathcal{L}_{JX}J)(Y) - J(\mathcal{L}_X J)(Y)\right).$$

This means that J is integrable if and only if the application on vector fields

$$X \mapsto \mathcal{L}_X J$$

is J-linear.

1.3. **Hermitian and Kähler metrics.** We now conisder a smooth manifold M equipped with an almost complex structure J and a riemannian metric g that is hermitian with respect to J in the sense that

$$g(JX, JY) = g(X, Y)$$

for all vectors X and Y.

Definition 1.3. We can define a skew-symmetric 2-form $\omega \in \Lambda^2_M$ by

$$\omega(X, Y) = g(JX, Y).$$

ω will be referred to as the *fundamental 2-form* or the *Kähler* form of g.

Since g is hermitian with respect to J, we can see that $J \cdot \omega = \omega$ which is to say that $\omega \in \Lambda^{1,1}_M$.

Let ∇^{LC} be the Levi-Civita connection for the metric g.

Definition 1.4. (M, J, g) is a *Kähler* manifold if

$$\nabla^{LC}(J) = 0.$$

Proposition 1.5. *(M, J, g) is Kähler if and only if*

- $N_J = 0$,
- $d\omega = 0$.

Proof. If $\nabla^{LC} J = 0$, since $\nabla^{LC}(g) = 0$, $\nabla^{LC}\omega = 0$. Since the connection is torsion free, this implies that $d\omega = 0$. Similarly, N_J can be expressed as a linear combination of terms of the form $\nabla_{\cdot} J(\cdot)$ so we have $N_J = 0$.

Conversely, one can verify the identity

$$g((\nabla_X J)Y, Z) = \frac{1}{2}\left(d\omega(X, Y, Z) - d\omega(X, JY, JZ)\right) + 2g(JX, N_J(Y, Z))$$

which gives that $d\omega = 0$ and $N_J = 0$ together imply that $\nabla^{LC} J = 0$.

We extend the inner product g on T_M by complex bilinearity to a non-degenerate symmetric bilinear form on $T_M \otimes \mathbb{C}$. The hermitian condition on g implies that

$$g(T^{1,0}, T^{1,0}) = 0,$$

and similarly for $T^{0,1}$. This is to say that $T^{1,0}$ and $T^{0,1}$ are isotropic subspaces for g. The inner product g does induce a hermitian product h on $T^{1,0}$ by

$$\begin{aligned} h(X^{1,0}, Y^{1,0}) &= 2g(X^{1,0}, \overline{Y^{1,0}}) \\ &= g(X, Y) - ig(JX, Y) \end{aligned}$$

where $X^{1,0} = \frac{1}{2}(X - iJX) \in T^{1,0}_M$.

The riemannian curvature of g is given by $R_{XY}Z = \left(\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]\right) Z$. Then, $\nabla J = 0$ implies that $R_{XY} \circ J = J \circ R_{XY}$, which implies that R_{XY} preserves the eigenspaces of J. That is,

$$R_{XY} : T^{1,0}_M \to T^{1,0}_M.$$

We have already seen that an endomorphism is a +1-eigenvector of J whenever it commutes with J, just as R_{XY} does. This notion is well-behaved with respect to raising and lowering indices using a hermitian metric in the sense that the 2-form given by

$$g(R_{XY}\cdot, \cdot)$$

is preserved by J. We can then say that $R_{XY} \in \Lambda^{1,1}$ and

$$R \in Sym^2(\Lambda^{1,1})$$

2. Lecture 2

2.1. **Hermitian Connections.** In this section we wish to compare the Chern and Levi-Civita connections on the tangent bundle to a Kähler manifold. For this, it is necessary to first give some definitions.

Given a holomorphic vector bundle $E \to M$ with a connection ∇, we can use the decomposition of forms to write $\nabla = \nabla^{1,0} + \nabla^{0,1}$ where

$$\nabla^{0,1} : \Omega^0(E) \to \Omega^{0,1}(E)$$
$$\text{satisfies} \quad \nabla^{0,1}(f\sigma) = \bar{\partial} f \otimes \sigma + f\nabla^{0,1}\sigma.$$

Definition 2.1. A $\bar{\partial}$-operator $\bar{\partial}^E$ is a differential operator

$$\bar{\partial}^E : \Omega^0(E) \to \Omega^{0,1}(E)$$
$$\text{such that} \quad \bar{\partial}^E(f\sigma) = \bar{\partial} f \otimes \sigma + f\bar{\partial}^E\sigma.$$

It is a standard statement of complex geometry that any holomorphic vector bundle admits a canonical $\bar{\partial}$-operator. If it is equipped with a hermitian product then there is a unique hermitian connection ∇ with $\nabla^{0,1} = \bar{\partial}^E$. Gauduchon shows that this in fact holds for arbitrary $\bar{\partial}$-operators on hermitian bundles over almost complex manifolds. There is no necessity for the square of the operator to be zero, or for the almost complex structure to be integrable. We state a less general version here.

Proposition 2.2. *Let (E, h) be a hermitian holomorphic vector bundle over a complex manifold M. Then there is a unique hermitian connection on E such that $\nabla^{0,1} = \bar{\partial}$.*

Proof. Let $\{e_i\}$ be a local holomorphic frame for E. Then any local section σ can be written as $\sigma = \sigma_i e_i$. Then, $\nabla\sigma = (d\sigma_i + \sigma_j\theta_{ji}) \otimes e_i$ where θ_{ji} is a matrix of one forms given by $\nabla e_j = \theta_{ji} \otimes e_i$. Then, if we assume that ∇ has $(0,1)$ part given by $\bar{\partial}$, $\nabla^{0,1}e_i = 0$ so θ_{ji} has type $(1,0)$. We set $h_{ij} = h(e_i, e_j)$. ∇ is a hermitian connection so

$$\begin{aligned} dh_{ij} &= \partial h_{ij} + \bar{\partial} h_{ij} \\ &= h(\nabla e_i, e_j) + h(e_i, \nabla e_j) \\ &= \theta_{ik} h_{kj} + h_{ik}\bar{\theta}_{jk}. \end{aligned}$$

Equating the parts of type $(1,0)$ we see that $\theta = \partial h \cdot h^{-1}$ which is determined exactly locally. One can verify that θ transforms so as to globally define a connection on M.

2.2. **The $\bar\partial$ operator on the holomorphic tangent bundle.** We return to (M,J,g) a complex manifold with a hermitian metric. The subspace $T^{1,0}_M \subseteq T_M \otimes \mathbb{C}$ is the $+i$ eigenspace of J. In local coordinates, $T^{1.0}$ is trivialised by

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \text{i.e.,} \quad J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}.$$

A local section of $T^{1,0}$ is given by $Z = Z_i \partial_{z_i}$ where Z_i are smooth local functions. The $\bar\partial$-operator is then defined on $T^{1,0}_M$ by

$$\bar\partial Z = (\bar\partial Z_i) \otimes \partial_{z_i} \in \Omega^{0,1}(T^{1,0}_M).$$

For real vectors $Y, Z \in T_M$ we can take the $(1,0)$ and $(0,1)$ parts

$$\begin{aligned} Y &\mapsto Y^{0,1} = \frac{1}{2}(Y + iJY) \in T^{0,1}_M \\ Z &\mapsto Z^{1,0} = \frac{1}{2}(Z - iJZ) \in T^{1,0}_M \\ \text{so } \bar\partial_Y Z &= (Y^{1,0} Z_i) \cdot \frac{\partial}{\partial z_i} \\ &= \left[Y^{1,0}, Z^{0,1}\right]^{1,0}. \end{aligned}$$

This can be identified with the real vector

$$\bar\partial_Y Z = 2\mathrm{Re}\left(\left[Y^{1,0}, Z^{0,1}\right]^{1,0}\right)$$

which coincides with

$$\bar\partial_Y Z = \frac{-1}{2} J\left(\mathcal{L}_Z J\right)(Y).$$

This last observation uses the fact that $N_J = 0$. This means that a real vector field $Z \in \Omega^0(T_M)$ is the real part of a holomorphic vector field if and only if $\mathcal{L}_Z J = 0$.

In the case that (M, J, g, ∇) is a Kähler manifold the Levi-Civita connection preserves the eigenspaces of J, and commutes with the projection of a real vector onto the $(1,0)$ and $(0,1)$ parts. That is, for example,

$$\nabla(Z^{1,0}) = (\nabla Z)^{1,0}.$$

The $(0,1)$ part of the connection is given by $\nabla^{0,1}_X Z^{1,0} = \nabla_{X^{0,1}} Z^{1,0}$ which has real part

$$\begin{aligned} \frac{1}{2}\mathrm{Re}\left(\nabla_{(X+iJX)}(Z - iJZ)\right) &= \frac{1}{2}\left(\nabla_X Z + \nabla_{JX} JZ\right) \\ &= \frac{-1}{2} J\left([Z, JX] - J[Z, X]\right) \\ &= \frac{-1}{2} J \circ (\mathcal{L}_Z J)(X) \end{aligned}$$

(Using the fact that ∇ is torsion-free). This shows that the $(0,1)$-part of the Levi-Civita connection is the canonical $\bar\partial$-operator on $T^{1,0}_M$.

We can summarise this quite simply in recalling that the Levi-Civita connection is uniquely determined as the unique torsion free connection on T_M that is compatible with the hermitian metric. The Chern connection is the unique connection on $T^{1,0}_M$ that is compatible with the metric and with $(0,1)$ given by the canonical $\bar\partial$-operator. The above calculations show that these two connections coincide if the metric is kählerian.

3. Lecture 3

3.1. **The Ricci and scalar curvatures.** We have previously seen that the riemannian curvature operator takes values in $Sym^2(\Lambda^{1,1})$. The Ricci curvature is defined by

$$\begin{aligned} r(X,Y) &= \operatorname{tr}\{Z \mapsto R_{XZ}Y\} \\ &= \langle R_{Xe_i}Y, e_i\rangle + \langle R_{XJe_i}Y, Je_i\rangle \end{aligned}$$

where $\varepsilon_j = 1/2(e_j - iJe_j)$ fors a hermitian orthonormal basis for $T^{1,0}$. The symmetry of R shows that

$$r(JX, JY) = r(X,Y)$$

which is to say that r is a hermitian symmetric form. Then if we define the *Ricci-form* of g to be

$$\rho(X,Y) = r(JX, Y)$$

then ρ is a skew-symemtric 2-form. The Ricci form is invariant under J so $\rho \in \Lambda^{1,1}$. A preliminary relationship between the Ricci and Kähler forms can be seen in the following calculation.

$$\begin{aligned} \rho(X,Y) &= \langle R_{JXe_i}Y, e_i\rangle + \langle R_{JXJe_i}Y, Je_i\rangle \\ &= \langle R_{X,Je_i}e_i, Y\rangle - \langle R_{Xe_i}Je_i, Y\rangle \\ &= \langle R_{e_iJe_i}X, Y\rangle + \langle R_{Xe_i}Je_i, Y\rangle - \langle R_{Xe_i}Je_i, Y\rangle \\ &= \langle R_{e_iJe_i}X, Y\rangle \\ &= R(\omega)(X \wedge Y). \end{aligned}$$

That is, $\rho = R(\omega) \in \Lambda^{1,1}$ where $\omega \in \Lambda^{1,1}$ and

$$R : \Lambda^{1,1} \to \Lambda^{1,1}.$$

The scalar curvature can also be dealt with in this way.

$$\begin{aligned} s = \operatorname{tr}_g(r) &= \langle r, g\rangle \\ &= \langle \rho, \omega\rangle \\ &= \langle R(\omega), \omega\rangle. \end{aligned}$$

In riemannian geometry, one can express the full curvature tensor as

$$R = \frac{s}{n(n-1)}Id + \frac{1}{n-2}r_0 \circ g + W \in \operatorname{End}(\Lambda^2, \Lambda^2)$$

involving the scalar, trace-free-Ricci and Weyl curvatures respectively. A similar expression exists for abstract curvature tensors of Kähler manifolds. This follows from the representation theory of $U(n)$ on $\Lambda^{1,1} \odot \Lambda^{1,1}$, with the condition that $\kappa(R) = 0$, where κ is the Bianchi map.

Let (M, J) be a complex manifold (i.e., $N_J = 0$) of dimension m. The *canonical bundle* is the holomorphic line bundle on M given by

$$K_M = \Lambda^{m,0} = \Lambda^m T_M^{*(1,0)}.$$

The anti-canonical bundle is the dual of K_M : $K_M^* = \Lambda^m T_M^{1,0}$. A hermitian metric h on $T^{1,0}$ induces a hermitian metric on K and K^*. If h is Kähler, K^* is preserved by the Levi-Civita connection of h and, since this coincides with the Chern connection on $T^{1,0}$, the induced connection on K^* is the Chern connection for this bundle and this metric.

Suppose that $\{\varepsilon_j = 1/2(e_j - iJe_j)\}$ is a local hermitian orthonormal frame for $T^{1,0}$. Then $\Phi = \varepsilon_1 \wedge \cdots \wedge \varepsilon_m$ locally spans K^*. The curvature of the induced connection on K^* is given by

$$\begin{aligned} R^{K^*}\Phi &= R^{K^*}\varepsilon_1 \wedge \cdots \wedge \varepsilon_m \\ &= \sum_j \varepsilon_1 \wedge \cdots \wedge (R\varepsilon_j) \wedge \cdots \wedge \varepsilon_m \\ &= \left[\sum_j h\left(R\varepsilon_j, \varepsilon_j\right)\right] \cdot \Phi \\ &= ig(R_{e_j Je_j}\cdot, \cdot)\Phi \\ &= i\rho\Phi \end{aligned}$$

That is, $R^{K^*} = i\rho$. This is one reason that Kähler geometry is so much more tractable that real riemannian geometry. This gives the Ricci curvature explicitly as the curvature of a geometrically defined line-bundle over the manifold. This calculation in particular shows that

(1) $d\rho = 0$
(2) $[\rho] \in 2\pi c_1(M, J) \in H^{1,1}_{dR}(M)$.

3.2. **The relation to the riemannian volume form.** Let $\{\varepsilon_\alpha = \frac{\partial}{\partial z_\alpha}\}$ be a local holomorphic frame for $T^{1,0}_M$. The metric is then locally given by the matix $g_{\alpha\bar{\beta}} = g(\varepsilon_\alpha, \overline{\varepsilon_\beta})$. The indeced metric on K^* is then given by

$$|\Phi|^2 = |\frac{\partial}{\partial z_1} \wedge \ldots \wedge \frac{\partial}{\partial z_m}|^2 = \det(g_{\alpha\bar{\beta}}).$$

Then the curvature of the Chern connection on K^* is given by

$$\begin{aligned} i\rho &= -\bar{\partial}\partial \log |\Phi|^2 \\ &= -\bar{\partial}\partial \log \det(g_{\alpha\bar{\beta}}). \end{aligned}$$

By contrast, the riemannian volume form is given by

$$\mathrm{d}vol_g = \frac{\omega^m}{m!} = \det(g_{\alpha\bar{\beta}})^{-1}\Omega \wedge \bar{\Omega}$$

where $\Omega = dz_1 \wedge \ldots \wedge dz_m$ locally trivialises $K_M = \Lambda^{m,0}_M$. This can be developed by comparing the volume forms for two Kähler metrics. If ω and ω' are two Kähler metrics (forms) on M and

$$\begin{aligned} {\omega'}^m &= F\omega^m \\ &= \frac{\det(g_{\alpha\bar{\beta}})}{\det(g'_{\alpha\bar{\beta}})} \cdot \omega^m \end{aligned}$$

then $i\bar{\partial}\partial \log F = \rho' - \rho$. That is, the difference of the Ricci forms of two Kähler metrics is directly by the (multiplicative) difference of the riemannian volume forms of the metrics.

3.3. **The Calabi conjecture.** In this section we will suppose that M is a compat complex manifold that admits a Kähler metric. We have seen that for any Kähler metric ω

$$\rho^\omega \in 2\pi c_1(M, J) \in H^2_{dR}(M).$$

We ask a converse question. Given $\rho \in 2\pi c_1(M, J)$ in the fixed cohomology class, is there a Kähler metric on M with ρ as its Ricci form? We refine this question to specify that the Kähler form can be taken within a chosen cohomology class.

Conjecture 3.1. *Let $\rho' \in 2\pi c_1(M,J)$. The for every $\alpha \in H^2_{dR}$ that contains a Kähler form, there exists a Kähler form $\omega' \in \alpha$ such that*

$$\rho^{\omega'} = \rho.$$

This statement can be refined further by recalling the functional dependence of the Ricci curvatures, and recalling the $\partial\bar{\partial}$-lemma. Suppose that ω' and ω are two Kähler forms with Ricci forms ρ' and ρ respectively. Then, $\rho' - \rho = i\bar{\partial}\partial \log F$. Then we can write $F = Ae^f$ where $\int f\omega^m = 0$. Then if the Kähler forms are cohomologous, their total volumes are equal, so

$$\int_M (\omega')^m = A \int_M e^f \omega^m$$

Conjecture 3.2. *Let (M,J,g) be a compact Kähler manifold. Let $f \in C^\infty(M,\mathbb{R})$ a smooth function and $A > 0$ a constant such that $A \int e^f \omega^m = vol(M)$. Then, there exists a Kähler form $\omega' \in [\omega]$ such that $(\omega')^m = Ae^f\omega^m$.*

That is to say that ever possible volume form is realised by some Kähler metric. We can see from the above that a specified volume form specifies f and A. f determines F, which gives the Ricci curvature of the new metric. By the dd^c-lemma, the Ricci curvature also specifies uniquely the new volume form.

By the dd^c-lemma, the condition $\omega' \in [\omega]$ is equivalent to

$$\omega' = \omega + i\partial\bar{\partial}\varphi$$

for some $\varphi \in C^\infty(M,\mathbb{R})$. We state a final equivalent version of this conjecture.

Conjecture 3.3. *Let (M,J,ω) be a compact Kähler manifold, $f \in C^\infty(M,\mathbb{R})$ and $A > 0$ such that*

$$A \int_M e^f \omega^m = \int_M \omega^m.$$

Then, there exists a unique $\varphi \in C^\infty(M,\mathbb{R})$ such that

(1) $\int_M \varphi\omega^m = 0$,
(2) $\omega' = \omega + i\partial\bar{\partial}\varphi$ *is a "positive"* $(1,1)$*-form,*
(3) $(\omega + i\partial\bar{\partial}\varphi)^m + Ae^f\omega^m$

The volume form of the new metric ω' is $Ae^f\omega^m$. The Ricci form of ω' is $\rho + i\bar{\partial}\partial f$. We can specify either of these data and find a Kähler metric in the given Kähler class.

We end this section by making note that this conjecture, in this final form, was proved by Yau in one of the landmark theorems of the twentieth century. In an incredible achievement of technical difficulty, he gave the foundational example of the use of hard non-linear analysis in geometry. A readable account of (a slightly different version of) the proof is to be found in the book of Joyce. The explanation given here of the sequence of equivalent conjectures was taken from that source.

4. Lecture 4

4.1. **The dd^c-lemma.** We recall that if J is an endomorphism of the vector space V such that $J^2 = -Id$ then J extends to $\Lambda^k V^*$ by

$$(J\alpha)(v_1, \ldots, v_k) = \alpha(J^{-1}v_1, \ldots, J^{-1}v_k).$$

If $T \in V \otimes V^*$, then $(J \cdot T)(v = JT(J^{-1}v)$ and $JT = T$ if and only if $T \in \mathfrak{gl}(V^{1,0}, \mathbb{C}) \subseteq \mathfrak{gl}(V, \mathbb{R})$. We define the differential operator

$$\begin{aligned} d^c &: \Omega^p \to \Omega^{p+1} \\ d^c\psi &= J \cdot d(J^{-1}\psi). \end{aligned}$$

we first note that this is a *real* operator, in contrast to ∂ and $\bar{\partial}$. It sends real differential forms to real differential forms. If $\psi \in \Omega^{r,s}$ is a form of type (r, s) then

$$\begin{aligned} J^{-1}\psi &= (-1)^{r+s} \cdot i^{r-s}\psi \\ d(J^{-1}\psi) &= (-1)^{r+s} i^{r-s} \left(\partial\psi + \bar{\partial}\psi\right) \\ Jd(J^{-1}\psi) &= -(i\partial\psi - i\bar{\partial}\psi) \end{aligned}$$

which gives that $d^c = i(\bar{\partial} - \partial)$ as a differential operator. In particular, on $\mathbb{C}$, for the real function $x = x(z)$, $d^c x = dy$. We can further understand other common operators in terms of d^c :

$$\begin{aligned} d &= \partial + \bar{\partial} \\ d^c &= i(\bar{\partial} - \partial) \\ \partial &= \frac{1}{2}(d + id^c) \\ \bar{\partial} &= \frac{1}{2}(d - id^c) \\ dd^c &= 2i\partial\bar{\partial}. \end{aligned}$$

On a hermitian manifold we wish to take the formal adjoints δ, δ^c of d, d^c such that, for example,

$$\int_M \langle d^c\alpha, \beta\rangle \mathrm{d}vol = \int_M \langle \alpha, \delta^c\beta\rangle \mathrm{d}vol$$

where α is a p-form and β is a $(p+1)$-form. If the manifold (M, J, g) is Kähler these operators are related by the *Kähler identities*

$$\begin{aligned} \delta = [\Lambda, d^c] &\qquad \delta^c = -[\Lambda, d] \\ d = [\delta^c, L] &\qquad d^c = -[\delta, L] \end{aligned}$$

where $L : \Lambda^{r,s} \to \Lambda^{r+1,s+1}$ is given by multiplication ω, and Λ is the pointwise adjoint of this map. Each of these four identities are equivalent. On$(\mathbb{C}^n, \omega_{euc})$ the observation $\delta = [\Lambda, d^c]$ can be seen relatively clearly. In general, the condition for a metric to be Käher is equivalent to the local existence of coordinates $\{z\}$ such that in these coordinates $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}} + h_{\alpha\bar{\beta}}$ where h vanishes to second order at a chosen central point.

Corollary 4.1. *Let (M, J, g) be a Kähler manifold. Then, $\delta d^c + d^c\delta = 0$.*

Proof. $\delta d^c = [\Lambda, d^c]d^c = -d^c\Lambda d^c$. The other term is similar.

For each of the operators d and d^c we can define Laplace operators

$$\begin{aligned} \Delta_d &= d\delta + \delta d \\ \Delta_{d^c} &= d^c\delta^c + \delta^c d^c. \end{aligned}$$

The first of these can be defined on any riemannian manifold. The second also uses the (almost) complex structure.

Corollary 4.2. *Let (M, J, g, ω) be a Kähler manifold. Then the Laplace operators for d and d^c coincide : $\Delta_d = \Delta_{d^c}$.*

This in particular means that the operators have the same kernel $\mathcal{H} \subseteq \Omega^p(X)$ and the same Green's operator

$$G = \Delta^{-1} \circ \Pi^{\perp}$$

where $\Pi^{\perp}$ is the orthogonal projection onto the complement of the kernel of Δ. This corollary gives us enhanced information on the Hodge decomposition. For $\psi \in \Omega^p(X)$ there exists a harmonic form ψ_H such that

$$\begin{aligned} \psi &= \psi_H + d\delta G\psi + \delta dG\psi \\ &= \psi_H + d^c\delta^c G\psi + \delta^c d^c G\psi \end{aligned} \tag{4.1}$$

with the same harmonic component and the same Green's operator.

Lemma 4.3. *Let (M, J, g) be a compact Kähler manifold. Then the Hodge decomposition of a real dd^c-closed p-form can be written as*

$$\psi = \psi_H + d\delta G\psi - d^c\delta\delta^c dG^2\psi.$$

Proof. $dd^c\psi = 0$, so when we differentiate Equation 4.1

$$\begin{aligned} 0 &= 0 + 0 + dd^c\delta dG\psi \\ &= -d\delta d^c dG\psi \\ &= d\delta dd^c G\psi \\ &\Rightarrow dd^c G\psi = 0 \end{aligned}$$

We then consider the d^c-Hodge decomposition for $dG\psi$.

$$\begin{aligned} dG\psi &= 0 + d^c\delta^c GdG\psi + \delta^c d^c GdG\psi \\ &= d^c\delta^c dG^2\psi - \delta^c Gdd^c G\psi \\ dG\psi &= d^c\delta^c G^2\psi \\ \Rightarrow \quad \psi &= \psi_H + d\delta G\psi + \delta d^c\delta^c dG^2\psi \\ &= \psi_H + d\delta G\psi - d^c\delta\delta^c dG^2\psi. \end{aligned}$$

Corollary 4.4. *let ψ be a real, J-invariant, d-exact p-form. Then $\psi = dd^c\tilde{\psi}$ for some $\tilde{\psi} \in \Omega^{p-2}$.*

Proof. We suppose by hypothesis that $\psi = d\phi$ and that $Jd\phi = d\phi$ implies that $dd^c\phi = 0$. We can then apply the above lemma to ϕ. We have that

$$\begin{aligned} \phi &= \phi_H + d\delta G\phi - d^c\delta\delta^c dG^2\phi \\ \text{and so } \psi = d\phi &= dd^c(-\delta\delta^c dG^2\phi). \end{aligned}$$

We now give the geometric application to Kähler and Ricci forms of Kähler metrics that we have desired.

Corollary 4.5. *Any two Kähler forms in the same cohomology class differ by dd^c of a real-valued function. The Ricci curvatures of any two Kähler metrics differ by dd^c of a real-valued function.*

5. Lecture 5

In this section we give the first non-trivial example of a Kähler manifold. This is the complex projective space. On this manifold practically everything can be calculated explicitly. We will give very concrete realisations of the tangent and canonical bundles, and show that it canonically admits a Kähler metric that is shown to have constant Ricci curvature. It can also easily be shown that this metric is symmetric, but this is a point of view that we will not pursue.

5.1. Complex projective space.

Let V be a complex vector space of dimension $m + 1$. We define $\mathbb{P}(V)$ denote the set of complex linear subspaces of V of dimension 1, said to be the set of complex lines in V. This manifold is called the complex projective space of dimension m. It can clearly be seen as the quotient

$$\mathbb{P}(V) = V^\times/\mathbb{C}^\times.$$

If we choose a basis $\{e^i\}$ for V, we can introduce homogeneous coordinates on $\mathbb{P}(V)$. That is, a line x spanned by the element $\bar{x} = u^i e^i$ is denoted by $[u_0; \dots : u_m]$.

We can take an atlas of charts for $\mathbb{P}(V)$ as follows. For each $\alpha \in V^* \setminus \{0\}$, set $\mathbb{P}^{(\alpha)}(V) = \mathbb{P}(V) \setminus \mathbb{P}(\ker(\alpha))$. This is the set of lines $x \subseteq V$ such that $\alpha|_x \neq 0$. This set is in a one-to-one correspondence with $\ker(\alpha) \cong \mathbb{C}^m \subseteq V$. To see this, we choose $v_\alpha \in V$ such that $\alpha(v_\alpha) = 1$ and define a map

$$\ker(\alpha) \ni w \mapsto x = \mathbb{C}(v_\alpha + w)$$

the image of which is a line in V. For two different $\alpha, \beta \in V^* \setminus \{0\}$, the transition maps for this identification can be seen to be holomorphic. This defines the structure of a complex manifold on $\mathbb{P}(V)$. $\mathbb{P}(V)$ is a compact complex manifod of dimension m.

If we choose a basis $\{e^i\}$ for V, we can introduce homogeneous coordinates on $\mathbb{P}(V)$. That is, a line x spanned by the element $\bar{x} = u_i e^i$ is denoted by $[u_0; \dots : u_m]$. If we take the covector $e_0^* \in V^*$, $\ker(e_0^* = \{(0, z_1, \dots, z_m)\}$. We can take $v_\alpha = e^0$ and $x \in \mathbb{P}^{e_0^*}$ can be expressed in homogeneous coordinates as

$$x = [1 : z_1 : \dots : z_m] = \mathbb{C}(e^0 + w)$$

where $w = (0, z_1, \dots, z_m)$.

5.2. The tautological line bundle.

We can canonically define a line bundle on $\mathbb{P}(V)$ as follows. At the point $x \in \mathbb{P}(V)$, considered as a line in V, we set

$$\begin{aligned} \Lambda_x &= \text{the line } x \subseteq V \\ \text{i.e., } \Lambda &= \{(x, u) \in \mathbb{P}(V) \times V \ ; \ u \in x\} \\ &\subseteq \mathbb{P}(V) \times V. \end{aligned}$$

For a given basis, $\{e^i\}$ for V with associated basis $\{e_i^*\}$ for V^* we consider the affine set $U_i = \mathbb{P}^{(e_i^*)}$. On this set Λ has a non-vanishing section

$$\sigma_i : [u_0 : \ldots : u_m] \mapsto \left(\frac{u_0}{u_i}, \ldots, 1, \ldots, \frac{u_m}{u_i}\right).$$

so Λ has holomorphic transition functions

$$\begin{aligned} g_{ij} \quad : \quad & U_i \cap U_j \to \mathbb{C}^\times \\ & [u_0, \ldots, u_m] \mapsto \frac{u_j}{u_i}. \end{aligned}$$

5.3. **The tangent and anti-canonical bundles.** We now ask whether we can identify the tangent bundle of $T\mathbb{P}^m$ in a geometric sense. We claim that for $x \in \mathbb{P}^m$, $T_x\mathbb{P}^m = x^* \otimes (V/x) = \mathrm{Hom}(x, V/x)$. This can be seen by first considering a hyperplane in V transverse to x such that $V = x \oplus P$. Then, for this decomposition, any line y *near* x is the graph of a map $A_y : x \to P$. That is,

$$y = \{\lambda + A_y(\lambda)\ ;\ \lambda \in x\}.$$

An infinitesimal version of this, independent of the choice of P, is that $T\mathbb{P}^m = \mathrm{Hom}(x, V/x)$.

The anti-canonical bundle is the next naturally defined bundle on $\mathbb{P}^m$.

$$\begin{aligned} K^*_{\mathbb{P}^m,x} = \Lambda^m T_x\mathbb{P}^m &= \Lambda^m (x \otimes V/x) \\ &= (x^*)^m \otimes \Lambda^m (V/x) \\ &\cong (x^*)^m \otimes x^* \otimes \Lambda^{m+1} V \\ &= (x^*)^{m+1} \otimes \Lambda^{m+1} V \end{aligned}$$

where the final factor is isomorphic to $\mathbb{C}$, a trivial bundle on $\mathbb{P}^m$. We conclude that $K^*_{\mathbb{P}^m} = \Lambda^{-(m+1)}$ as line bundles on $\mathbb{P}^m$.

5.4. **The Fubini-Study metric.** There exists a fibration $S^1 \to S^{2m+1} \to \mathbb{P}^m$ where we consider the sphere in $\mathbb{C}^{m+1}$ and S^1 acts by complex multiplication. Let T be the vector field on S^{2m+1} tangent to the S^1-orbits and η the 1-form dual to T. Set $\mathcal{H}_u = \ker(\eta)$ at $u \in S^{2m+1}$. This is a complex hyperplane in $\mathbb{C}^{m+1}$ and projects isomorphically onto the tangent space to $T\mathbb{P}^m$. That is, $S^{2m+1} \to \mathbb{P}^m$ defines an S^1-principal bundle over $\mathbb{P}^m$ with associated line bundle Λ. η and $\mathcal{H}$ define a connection on this bundle.

$\pi : \mathcal{H} \to T_x\mathbb{P}^m$ is an isomorphism, so we can define a metric on $\mathbb{P}^m$ by setting

$$g_{FS}(X, Y) = g_0(X_H, Y_H)$$

where X_H and Y_H are the horizontal lifts of X and Y respectively, and g_0 is the round metric on S^{2m+1}. The metric defines a hermitian form $\omega_{FS} = g_{FS}(J\cdot, \cdot)$ and this can be seen to satisfy

$$d\eta = \pi^*\omega_{FS}. \tag{5.1}$$

This can be seen in a number of ways. We recall the formula for vector fields

$$d\eta(X, Y) = \frac{1}{2}\left(X\eta(Y) - Y\eta(X) - \eta([X, Y])\right)$$

This reduces the calculation to the case that the vector fields are vertical or horizontal. We recall that

$$S^{2m+1} = SU(m+1)/SU(m)$$

where the action of $SU(m+1)$ preserves the distribution $\mathcal{H} = \ker(\eta)$. This further reduces the calculation even further, to that of a plane field on S^3. Claudio Gorodski pointed out that this observation can be seen even more clearly, by expressing η explicitly in coordinates from $\mathbb{C}^{m+1}$. The derivative then coincides with ω_{euc}, which restricts to $\mathcal{H}$ to be the pull-back of ω_{FS}.

The above has a number of immediate consequences.

(1) $d\omega_{FS} = 0$. This follows from Equation 5.1.
(2) $R^\Lambda = d\eta = -\omega_{FS}$. This follows from the observation that η defines the natural Chern connection on Λ. This means that Λ is a *negative* line bundle.
(3) $R^{K^*} = \rho_{FS} = -(m+1)R^\Lambda = (m+1)\omega_{FS}$. That is, $\rho_{FS} = (m+1)\omega_{FS}$ which is to say that ω_{FS} is a Kähler-Einstein metric . This equation follows from the explicit calculation of K^* in terms of Λ.

In particular, we can obtain an explicit expression for ω_{FS} in terms of local holomorphic coordinates on $\mathbb{P}^m$. We have

$$-\omega = R^\Lambda \quad = \quad i\bar{\partial}\partial\log(h)$$

where $h = h(\sigma)$ is the length of a local non-zero holomorphic section. In this case, on $\{u_0 \neq 0\} = \{[1 : z_1 : \ldots : z_m]\}$, we can take

$$\begin{aligned} \sigma([u]) &= (1, z_1, \ldots, z_m) \\ \omega &= -i\bar{\partial}\partial\log(1 + \sum |z_i|^2) \\ &= dd^c\log(1 + \sum |z_i|^2). \end{aligned}$$

6. Lecture 6

6.1. **The blow-up construction.** We now consider an algebro-geometric construction for obtaining new complex manifolds. We first recall the tautological bundle on $\mathbb{P}(V)$. Suppose that V is a complex vector space of dimension m, so that $\mathbb{P}(V)$ is of dimension $m-1$. The bundle Λ on $\mathbb{P}(V)$ is defined by

$$\Lambda = \{(u, x)\ ;\ u \in x\} \subseteq V \times \mathbb{P}(V).$$

The bundle projection π of a point in Λ is given by the projection to the second factor here. Λ is then a complex submanifold of $V \times \mathbb{P}(V)$ of dimension m. Consider instead the projection to the first factor $p : \Lambda \to V$, $(u, x) \mapsto u$. We observe :

(1) For $u \neq 0 \in V$ the point $x = [u]$ such that $u = p(u, x)$ is defined uniquely.
(2) For $u = 0 \in V$, $0 \in x$ for all $x \in \mathbb{P}(V)$

which is to say that $\Lambda \to V$ is surjective. It is an isomorphism away from $0 \in V$ and $p^{-1}(0) \cong \mathbb{P}^{m-1}$ is equal to the image of the zero-section in Λ. This construction, the act of replacing V by Λ is what we mean by blowing up $0 \in V$. This can be done to any point in a complex manifold M. That is, take a neighbourhood bilholomorphic to $B(0, \varepsilon) \subseteq V$. Replace this set with the set $\{(u, x); |u| < \varepsilon\} \subseteq \Lambda$. This surgery replaces a point $p \in M$ with the projective space $\mathbb{P}(T_pM)$.

6.2. **The Hirzeburuch surface.** In this section we present an example of the blow-up of a manifold, in a case where we can explicitly recognise the resulting manifold. This allows us in particular to identify a large set of Kähler metrics on this manifold. To start, we state explicitly that we will blow-up a point in a complex projective space $\mathbb{P}(V)$.

Let $x_0 \in \mathbb{P}(V)$, where $V \cong \mathbb{C}^{m+1}$. Let Q_{x_0} denote the set of projective lines in $\mathbb{P}(V)$ that pass through x_0. This can be seen to be isomorphic to $\mathbb{P}^{m-1}$. A projective line will be considered $\mathbb{P}(E)$ where $E \subseteq V$ is a 2-dimensional linear subspace. If D contains x_0, then D is uniquely determined by specifying a line y in V transverse to x_0 so that $D = \mathbb{P}(x_0 + y)$. We consider the locus of an incidence relation involving Q_{x_0}.

$$\widehat{\mathbb{P}(V)}_{x_0} = \{(x, D) \in \mathbb{P}(V) \times Q_{x_0} \ ; \ x \in D\}, \pi : \widehat{\mathbb{P}(V)}_{x_0} \to \mathbb{P}(V).$$

Then, just as with the blow-up of a point in V, we see that

(1) π is an isomorphism away from x_0 (there is a unique projective line between x and x_0).
(2) $\pi^{-1}(x_0) = \{x_0\} \times Q_{x_0} \cong \mathbb{P}^{m-1}$.

Intuitively, the blow-up separates all of the projective lines that pass through x_0. We can define $\widehat{\mathbb{P}(V)}_{x_0}$ explicitly as a fibre bundle. Let V_0 be a hyperplane (codimension 1 linear subspace) of V such that $V = x_0 + V_0$. Then,

$$\begin{array}{rcl} Q_{x_0} & \cong & \mathbb{P}(V_0) \cong \mathbb{P}^{m-1} \\ D & \mapsto & D \cap \mathbb{P}(V_0). \end{array}$$

This map essentially acts as a linear projection of $\mathbb{P}(V) \setminus \{x_0\}$ onto $\mathbb{P}(V_0)$. Let Λ^{V_0} be the tautological bundle on $\mathbb{P}(V_0)$. That is,

$$\Lambda^{V_0} = \{(u, x) \ ; \ u \in x\} \subseteq V_0 \times \mathbb{P}(V_0).$$

Proposition 6.1.

$$\widehat{\mathbb{P}(V)}_{x_0} = \mathbb{P}(1_{x_0} + \Lambda^{V_0}).$$

In this proposition, we have that 1_{x_0} is a trivial line bundle on $\mathbb{P}(V_0)$ with fibre the line x_0. $1_{x_0} + \Lambda^{V_0}$ is a rank-2bundle on $\mathbb{P}(V_0)$ and $\mathbb{P}(1_{x_0} + \Lambda^{V_0})$ is the projectivised bundle with fibre $\mathbb{CP}^1$. Let $p : \mathbb{P}(1_{x_0} + \Lambda^{V_0}) \to \mathbb{P}(V_0)$ denote the projection map of the fibration.

Proof. This proposition is almost tautological. For $(x, D) \in \widehat{\mathbb{P}(V)}_{x_0}$, $y = D \cap \mathbb{P}(V_0)$ is well-defined. Considering y as a line in V we have that $x \in D = \mathbb{P}(x_0 + y)$ which is the fibre of $\mathbb{P}(1_{x_0} + \Lambda^{V_0})$ at y. Conversely, a point in this fibre determines a point of the form (x, D) where $x \in D$.

That is, $\widehat{\mathbb{P}(V)}_{x_0}$ is the blow-up of $\mathbb{P}(V)$ at x_0 but can also be identified as a $\mathbb{P}^1$-bundle over a lower dimensional projective space. The blow-up admits an *exceptional hypersurface* or divisor, given by $\pi^{-1}(x_0)$. This can be recovered from the fibration model as well. The exceptional divisor is given by $\{x_0\} \times Q_{x_0} \subseteq \widehat{\mathbb{P}(V)}_{x_0}$. The fibration $\mathbb{P}(1 + \Lambda) \to \mathbb{P}(V_0)$ admits two sections, σ_0 and σ_∞. These correspond to the points at $0, \infty \in \mathbb{P}^1$ but can also be expressed as, for $y \in \mathbb{P}(V_0)$,

$$\begin{array}{rcl} \sigma_\infty : y & \mapsto & y \subseteq x_0 + y \\ \sigma_0 : y & \mapsto & x_0 \subseteq x_0 + y. \end{array}$$

The images of these sections define smooth submanifolds Σ_∞ and Σ_0 of $\widehat{\mathbb{P}(V)}_{x_0}$. Via the identification of Q_{x_0} and $\mathbb{P}(V_0)$, the image of σ_0 then corresponds to the exceptional divisor, described above.

We can then consider a family of Kähler metrics on $\widehat{\mathbb{P}}$. Let ω_V denote the Fubini-Study metric on $\mathbb{P}(V)$ and ω_{V_0} denote the Fubini-Study metric on $\mathbb{P}(V_0)$. We then consider the 2-form on $\widehat{\mathbb{P}}$

$$\omega_{a,b} = a\pi^*\omega_V + bp^*\omega_{V_0}$$

for $a, b > 0$. It is clear that the two summands are semi-positive forms. The projection π is an isomorphism away from the exceptional locus $\pi^{-1}(x_0)$, which is identified with Σ_0. However, Σ_0 is a section of the fibration p, so $p^*\omega_{V_0}$ is strictly positive tangent to Σ_0.

We then conclude that $\omega_{a,b}$ form a family of Kähler metrics on $\widehat{\mathbb{P}(V)}_{x_0}$.

7. Lecture 7

7.1. **The Eguchi-Hanson metric.** The previous examples were of compact homogeneous Kähler manifolds. The projective spaces were seen to be Kähler-Einstein manifolds with positive Einstein constant. In this section we will construct a Ricci-flat (hence Einstein) Kähler metric on a non-compact manifold that is asymptotically flat. This metric was originally constructed by the physicists Eguchi and Hanson as an example of a gravitational instanton, that being in our terminology a non-compact space with finite total curvature, asymptotic to the euclidean metric and with anti-self-dual Weyl curvature. We present a slightly different construction, that the author became aware of from the article [**?**] of Donaldson but was well-known earlier.

The Eguchi-Hanson has other properties that show it as an example of many more general constructions. The fact that it is simply-connected and Ricci-flat Kähler and 4-dimensional means that it is in fact a hyperkähler manifold. As such, it has been generalised by Hitchin and Kronheimer using the hyperkähler quotient construction to give other ALE metrics on resolutions of quoient singularities. The metric is defined on the total space of a line bundle. Kähler-Einstein metrics on the total spaces of bundles were constructed by Calabi, including this metric among them. Other bundle constructions of Einstein metrics were done by Gibbons, Page and Pope, and Bryant and Salamon.

We recall that if (X, ω) is a complex manifold of (complex) dimension 2 and ω a Kähler metric, and if $\chi \in \Omega^{2,0}$ is a local holomorphic trivialisation, then the Ricci curvature is given by

$$\rho = i\partial\bar{\partial}\log|\chi|^2.$$

That is, the Ricci form is the curvature of the anti-canonical bundle K^*. In particular, if we have the relation

$$\omega^2 = \lambda\chi \wedge \bar{\chi}$$

for some constant λ, then $Ric(\omega) = 0$. Here, $\lambda = |\chi|^{-2}$ constant implies that $\rho = i\partial\bar{\partial}\log|\chi|^2 = 0$.

On $\mathbb{C}^2$ we can take the holomorphic form $\chi = dz_1 \wedge dz_2$. We want to find a Kähler form ω such that $\omega^2 = \lambda\chi \wedge \bar{\chi}$. In particular, we make the assumption that the metric ω is rotationally symmetric and that $\omega = i\partial\bar{\partial}F$ where $F = F(\rho)$ for $\rho = |z_1|^2 + |z_2|^2$ (radius squared). We will determine the F so that $Ric = 0$.

For example, for $F(\rho) = \rho = |z_1|^2 + |z_2|^2$, $\omega = dz_i \wedge d\bar{z}_i$, i.e., the euclidean metric.

$\omega = i\partial\bar{\partial}F$ can be expressed as

$$\begin{aligned} \omega &= i\partial\bar{\partial}F \\ &= \begin{pmatrix} dz_1 & dz_2 \end{pmatrix} \begin{pmatrix} F' + |z_1|^2 F'' & z_1 \bar{z}_2 F'' \\ \bar{z}_1 z_2 F'' & F' + |z_2|^2 F'' \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \end{pmatrix} \end{aligned}$$

The condition that $\omega^2 = \lambda \chi \wedge \bar{\chi}$ gives the equation

$$\det \begin{pmatrix} F' + |z_1|^2 F'' & z_1 \bar{z}_2 F'' \\ \bar{z}_1 z_2 F'' & F' + |z_2|^2 F'' \end{pmatrix} = \lambda$$

which simplifies to $F'(F' + \rho F'') = \lambda$. We suppose that $\lambda = 1$. This can first be solved for F' by

$$F'(\rho) = \sqrt{1 + \left(\frac{A}{\rho}\right)^2}$$

for some $A \in \mathbb{R}$. This reduces to two cases : $A = 0$ and $A \neq 0$. In the first case $F' = 1$ and $F = \rho$ gives the euclidean metric. In the case $A \neq 0$ we set $A = 1$. The substitution $\rho^{-1} = \sinh(t)$ can then be used to solve the equation for F,

$$F(\rho) = \sqrt{\rho^2 + 1} + \ln(\rho) - \ln\left(\sqrt{\rho^2 + 1} + 1\right)$$

In particular, $F'(\rho) = 1 + O(r^{-4})$ for large r implies that $\omega = i\partial\bar{\partial}F = \omega_{euc} + O(r^{-4})$ for $\rho = r^2 >> 1$, together with all of its derivatives. This is what we mean by asymptotically locally euclidean.

We have that $\omega \to \omega_{euc}$ as $r \to \infty$, but $F \to -\infty$ as $r \to 0$. A priori, ω does not extend across 0. It is necessary to topologically change the manifold so as to get a smooth extension. We recall that $F = F(\rho)$ so ω is $U(2)$-invariant. In particular, ω descends to a metric on $\mathbb{C}^2/\mathbb{Z}_2$, away from the origin. We claim that ω extends to a Ricci-flat metric on $T^*\mathbb{P}^1$, which is a desingularisation of $\mathbb{C}^2/\mathbb{Z}_2$.

We recall that at $x \in \mathbb{P}^1$,

$$\begin{aligned} T_x\mathbb{P}^1 &= x^* \otimes \mathbb{C}^2/x \\ T_x^*\mathbb{P}^1 &= x \otimes \mathrm{Ann}(x) \\ \mathrm{Ann}(x) &= \{\alpha \in \mathbb{C}^{2\,*} \ ; \ \alpha|_x = 0\}. \end{aligned}$$

Then for $\Omega = dz_1 \wedge dz_2$ on $\mathbb{C}^2$ we can define the map

$$\begin{aligned} \pi : \mathbb{C}^2 \setminus \{0\} &\to T^*\mathbb{P}^1 \\ v &\mapsto v \otimes \Omega(v, \cdot) \in T_x^*\mathbb{P}^1 \end{aligned}$$

for $x = [v]$. This map is

(1) non-linear
(2) surjective onto $T^* \setminus \{0 - \text{section}\}$
(3) holomorphic
(4) $2:1$.

This means that

$$\mathcal{T}^*\mathbb{P}^1 \setminus \{0 - \text{sect.}\} \cong \mathbb{C}^{2\times}/\mathbb{Z}_2.$$

On $\mathbb{CP}^1$ we take $x = [u_0 : u_1] \subseteq U_0 = \{u_0 \neq 0\}$. I.e., $x = [(1, z)]$ where z is a local co-ordinate. Then, dz is a local non-zero 1-form in $\Omega^{1,0}(U_0, T^*)$.

Lemma 7.1.

$$dz = v \otimes \Omega(v, \cdot)$$

at $x = [1 : z]$ *for* $v = (1, z) \in \mathbb{C}^2 \setminus \{0\}$.

Proof. This is quite easy to show, once the correct identifications are made. Since $T^{1,0}$ is 1-dimensional, this can be seen by evaluating the two sides on a given $(1, 0)$ vector, such as $\frac{\partial}{\partial z}$. Both sides evaluate to 1, showing that they are equal.

Local coordinates on $T^*\mathbb{P}^1$ are given by

$$\begin{aligned} (z, \lambda) &\mapsto \lambda dz \\ &= \lambda(1, z) \otimes \Omega((1, z), \cdot) \\ &= (\lambda^{1/2}) \otimes \Omega\left(\lambda^{1/2} v, \cdot\right). \end{aligned}$$

The function ρ (from $\mathbb{C}^2$) in these coordinates is

$$\rho = |\lambda^{1/2} v|^2 = |\lambda| \left(1 + |z|^2\right).$$

Then, to show that the metric $\omega = \imath\partial\bar{\partial}F$ extends across the 0-section in $T * \mathbb{P}^1$ we consider the function

$$F(\rho) = \sqrt{\rho^2 + 1} + \ln(\rho) - \ln\left(\sqrt{\rho^2 + 1} + 1\right).$$

For example, for $\rho = |\lambda|(1 + |z|^2)$,

$$i\partial\bar{\partial} \ln \rho = i\partial\bar{\partial} \ln |\lambda| + i\partial\bar{\partial} \ln(1 + |z|^2).$$

The first term on the right can be seen to be identically 0. The second term coincides with the Fubini-Study metric in these coordinates.

The other terms can also be calculated. It can be seen that they are identically zero in the directions tangent to the 0-section, and positive (for example, a multiple of $(1 + |z|^2) d\lambda \wedge d\bar{\lambda}$) in the fibre directions. This shows that the metric extends across the 0-section of $T^*\mathbb{P}^1$.

This completes this section on the Eguchi-Hanson metric, and so of these notes.

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